Market-Clearing Electricity Prices and Energy Uplift

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Electricity market models require energy prices for balancing, spot and short-term forward transactions. For the simplest version of the core economic dispatch problem, the formulation produces a well-defined solution to the energy pricing problem in the usual form of the intersection of the supply marginal cost curve and the demand bids. In the more general economic unit commitment and dispatch models, there may be no analogous energy price vector that is consistent with and supports the quantities in the economic dispatch solution. Uplift or make-whole payments arise in this condition. Comparison of three alternative pricing models illustrates different ways to define and calculate uniform energy prices and the associated impacts on the energy uplift required to support the least cost unit commitment and dispatch.

Introduction

Electricity markets require energy prices for balancing, spot and short-term forward transactions. In the core framework of bid-based-security-constrained-economic-dispatch, locational energy prices that are consistent with market equilibrium appear as the locational system marginal costs associated with the economic dispatch solution. These prices are charged to loads and paid to suppliers. The prices support the equilibrium solution in the sense that at these prices competitive suppliers and loads would have no incentive to change their bids and would have an incentive to follow the dispatch. However, the core model does not incorporate all the features of the electricity market, such as the discrete nature of unit commitment. Going beyond the simple core model, there are important cases where there are no exact prices that support the quantities determined in the economic electricity dispatch, i.e., there are no exact prices that support the economic equilibrium.

In practice, two general types of problems arise in choosing and using electricity prices. First, there may be products and services that are not included in the formal model, so that the model does not produce market prices for these products in the form of marginal costs. For example, reactive power may not be formally represented in the model so that there are no reactive power prices.[1] Hence, reactive power and similar ancillary services must be addressed outside the formal model. Second, even when products are included in the formal model, there may be no set of prices for these products that support the dispatch solution. For instance, the core model assumes all the decision variables are continuous and, because of this, it is relatively easy to identify the

[1] Including reactive power explicitly in the model of dispatch is possible in principle, but it is not yet common practice. W. Hogan, "Markets in Real Electric Networks Require Reactive Prices," Energy Journal, Vol.14, No.3, 1993.

optimal solution in terms of both the optimal dispatch quantities and the associated prices. But for certain critical choices in the unit commitment stage, the relevant decisions are discrete and not continuously variable. The individual plant is either off or is committed, a zero-one choice. In addition, the actual dispatch may be nearly but not exactly optimal because of software limitations or operator intervention to address specific constraints not accounted for in the formal model. Both discrete decision variables and less than fully optimal solutions can produce circumstances where there are no exact prices that support the electricity dispatch.

The discussion and analysis here address the second of these two types of problems, in which exact prices do not exist to support the dispatch solution because the underlying problem contains discrete decisions, or because the solution is nearly but not exactly optimal. When there is no set of energy prices that supports the solution, this requires some accommodation in selecting a workable rule for pricing electric energy and treating the implications for any deviation from the equilibrium solution. The first half of the paper develops the general interpretation with accompanying graphical illustrations. The second half of the paper summarizes a formal model.

Market-Clearing Prices

Before considering the details of a representative electricity commitment and dispatch model, a more general statement of the issues in terms of the fundamentals of constrained optimization highlights the critical ideas and issues related to pricing. We state the basic issue in terms of the fundamentals of constrained optimization, and later extend the concepts to the electricity market model as a special case.

Consider a generic optimization problem for fixed y,

$$\begin{aligned} &\underset{x \in X}{Min} && f(x) \\ &s.t. && g(x) = y. \end{aligned}$$

The decision variables in the vector x must meet two types of constraints. First, the decision variables must be in the set X, which may incorporate many different constraints or special characteristics. For example, part of the specification of X may be that some or all the variables take on discrete values.

The vector y represents some external requirement that must be met by the choice of the decision variables, x. For instance, in the economic dispatch problem the external requirement might be to meet a certain level of net demand. The constraint functions $g(x)$ map the commitment and dispatch decision variables into the external requirement to meet load y. The separation of the $g(x)$ constraints from those in the set X implicitly recognizes that there are instances in which the constraints $g(x)$ are complicating and the optimization problem would be easy to solve if these constraints could be removed. For example, in the electricity unit commitment and economic dispatch problem we might have the constraints in $g(x)$ describing interactions across many separate unit decisions, while the set X decomposes into many separable and individually simple problems.

In the broad context of the optimization literature, the general objective function $f(x)$ could be some form of cost-benefit function. The electricity market interpretation might be that this is the cost of commitment and dispatch to meet load. A more general formulation would see the objective function as some cost-benefit aggregation. For the present purposes it is simpler to explain the concepts using the interpretation of minimizing costs to meet load.

As the load varies, the value of the least-cost solution changes accordingly. Define the value function (a.k.a., minimum cost function, perturbation function, auxiliary function) as

$$v(y) = \underset{x \in X}{Inf} \left\{ f(x) \middle| g(x) = y \right\}.$$

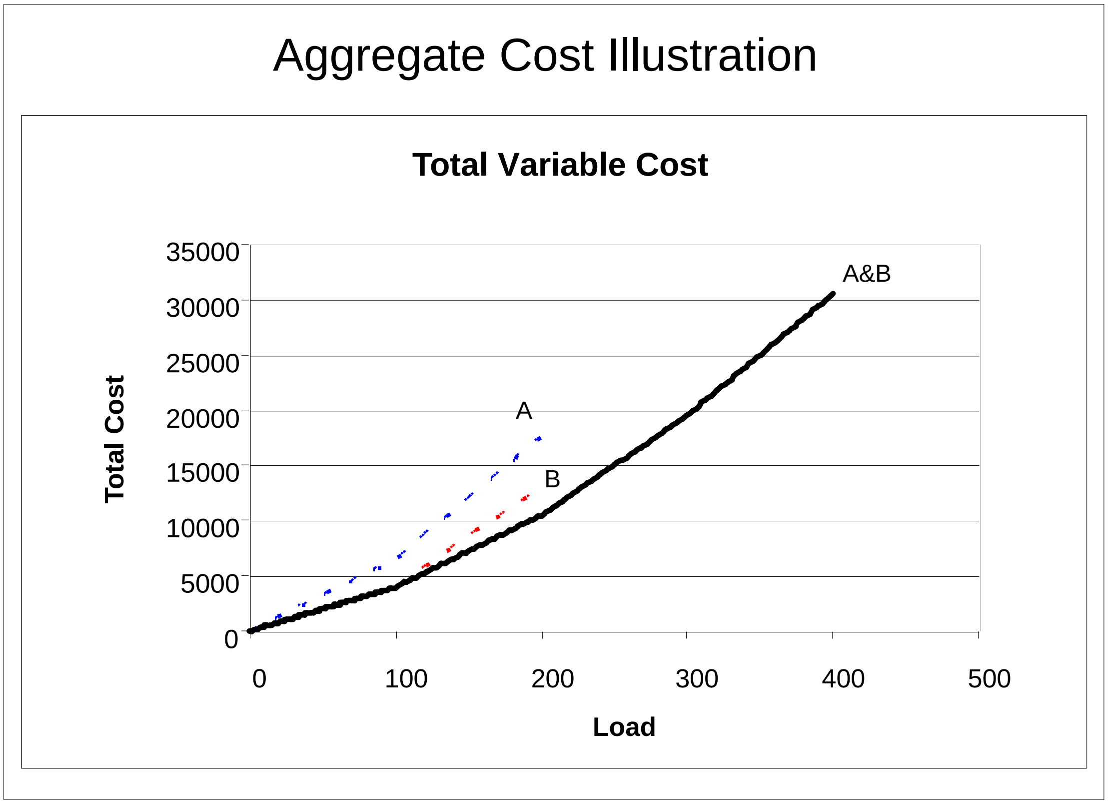
The value function plays a central role in the definition and derivation of equilibrium prices. The slope of the value function represents the marginal cost of meeting an additional unit of load. Along with the more general derivation when both load and generation are optimized, this marginal cost defines the market-clearing price. [2]

The requirements for this interpretation of the marginal cost as the appropriate price are met by the core electricity dispatch model with continuous dispatch variables and well-behaved cost functions. For example, consider three plants as shown in the table. The first illustration utilizes just the first two generating units, each with two levels of variable cost for up to a 100 MW each for a total capacity of 400 MW.

		Plants		
	q (MW)	A	B	C
Fixed Cost ($)		0	6000	8000
Var cost1 ($/MWh)	100	65	40	25
Var cost2 ($/MWh)	100	110	90	35

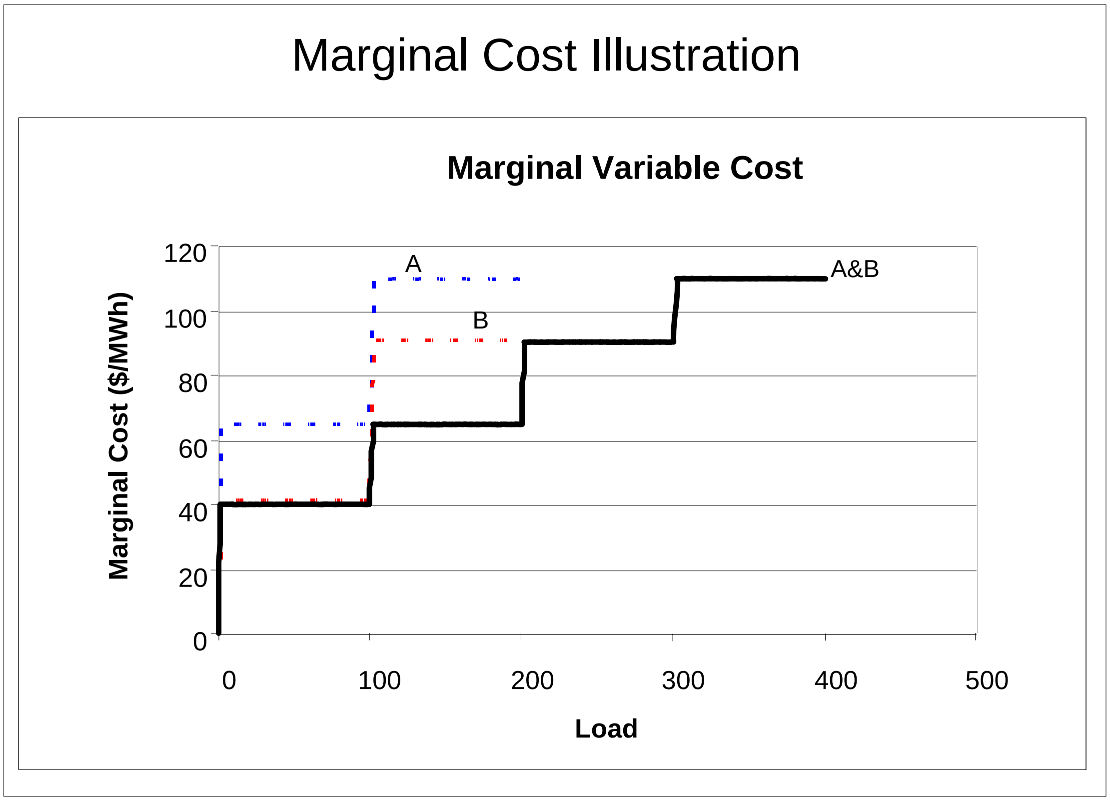
In the first illustration, we also ignore the fixed cost of committing the plants, and take into account only the variable costs, so that all of the dispatch variables are continuous. The individual and aggregate least total cost for each level of load—i.e., the value function—for this example would be as shown in the figure.

[2] For a further discussion of market equilibrium concepts, see William W. Hogan and Brendan R. Ring, "On Minimum-Uplift Pricing for Electricity Markets," March 19, 2003, (available at http://ksghome.harvard.edu/~WHogan/minuplift_031903.pdf).

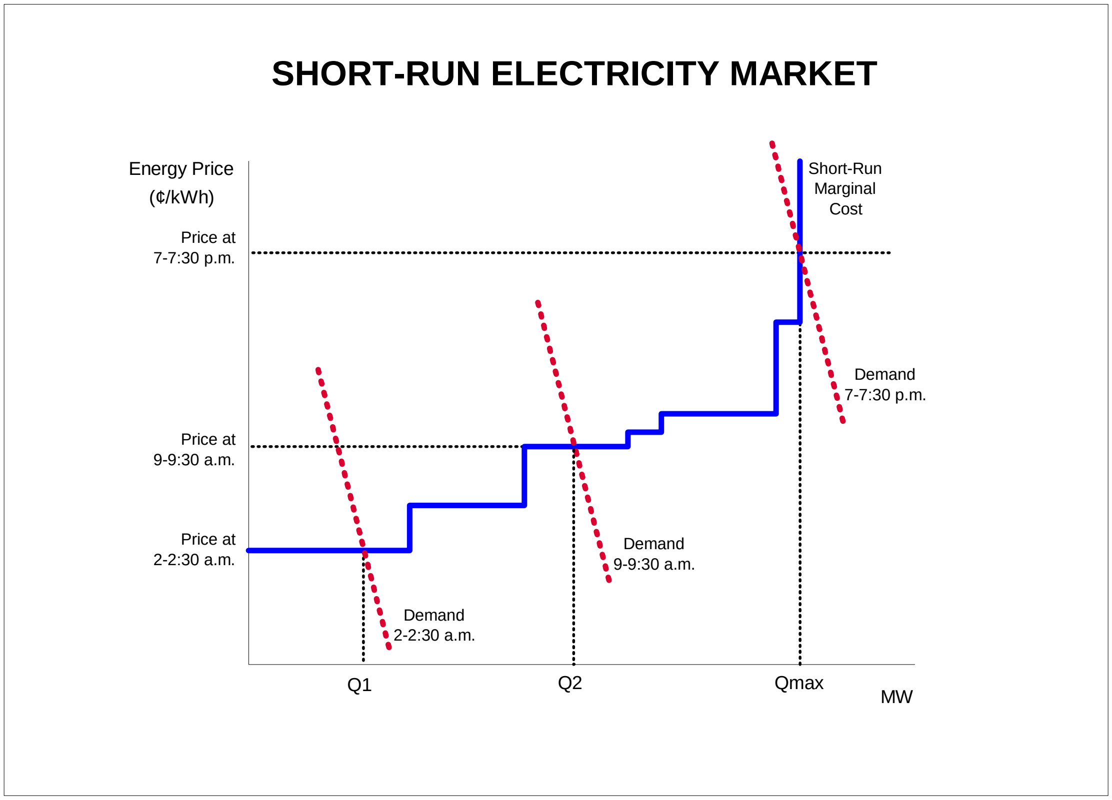


Here the first level of variable cost for plant B is the cheapest, and defines the aggregate cost curve up to 100 MW. The succeeding increments in the total cost function follow in order of increasing variable cost. A more familiar way of showing the same information would be in the individual and aggregate marginal supply curves, which show the marginal cost of each unit of output, as a function of increasing output, instead of showing the aggregate cost. When the variable costs are constant over ranges, as in this example, the resulting marginal supply function consists of a series of steps. The vertical steps are the points where the aggregate cost function is not differentiable, and any price within the vertical segment would support the quantity dispatch solution. These vertical segments present some technical issues to be ignored here as not important to the main discussion.

The aggregate supply function is the horizontal sum of the supply functions of individual generating units. This supply function could be cost-based, or the variable cost of generator operation, or could consist of the supply offers in a coordinated electricity market.

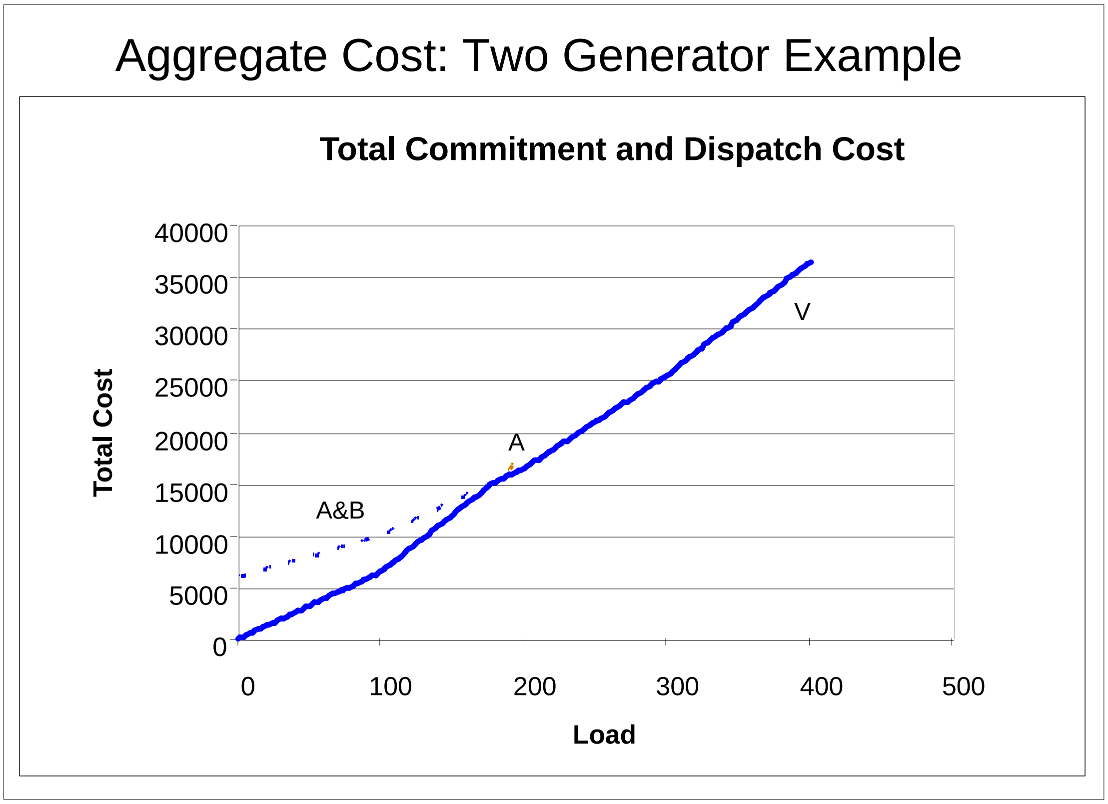


For any given level of load, the aggregate supply function determines the market clearing price at the level of demand or the point of intersection with the demand curve. In the electricity market model, these prices are the market clearing prices that satisfy the "no arbitrage" condition that there are no remaining profitable trades among the market participants.

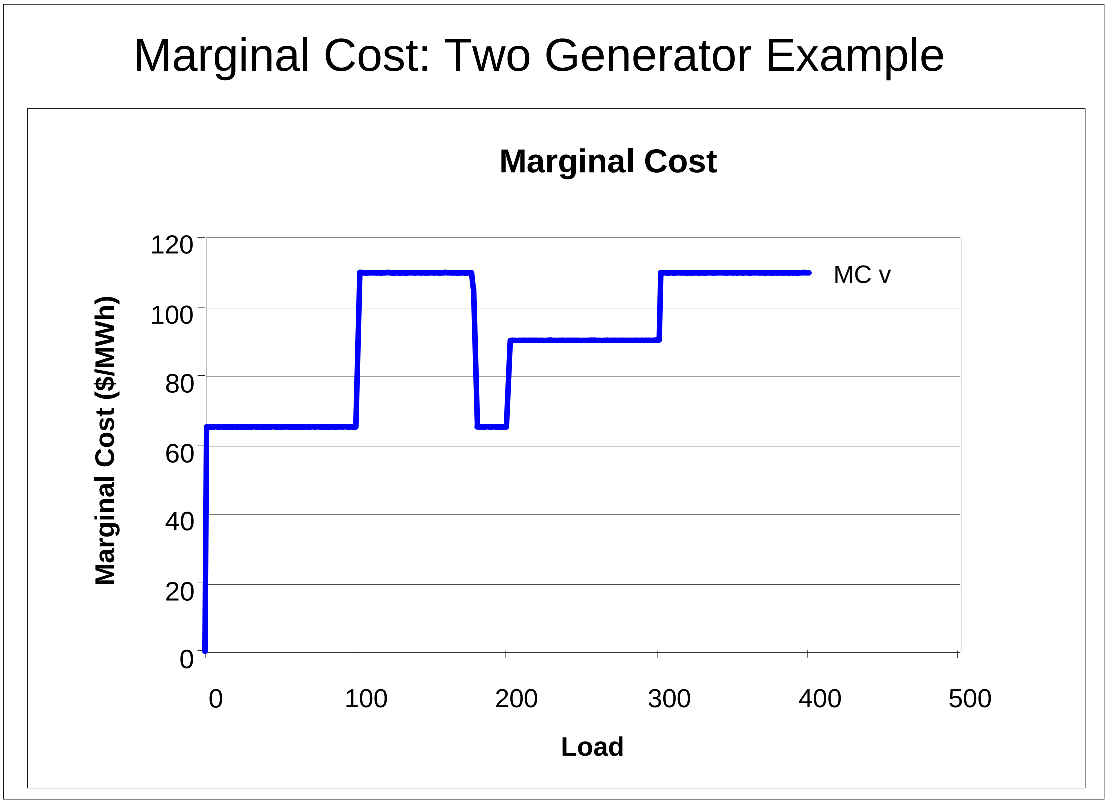


Adding more plants and more steps, allowing for offers that are not step functions, embedding in a transmission system, and so on, present no conceptual difficulties. The generalization of the supply function applies, with increasing load giving rise to increasing prices. With offers equal to variable costs, the resulting prices are incentive compatible in the sense that given the prices no participant would wish to change its offer or change the level of supply.

Moving beyond this core model, the next step is to include the fixed cost of plant B and consider that it would not be optimal to commit plant B until the level of load was high enough to absorb these fixed costs. This would produce a more complicated aggregate cost picture, as in:

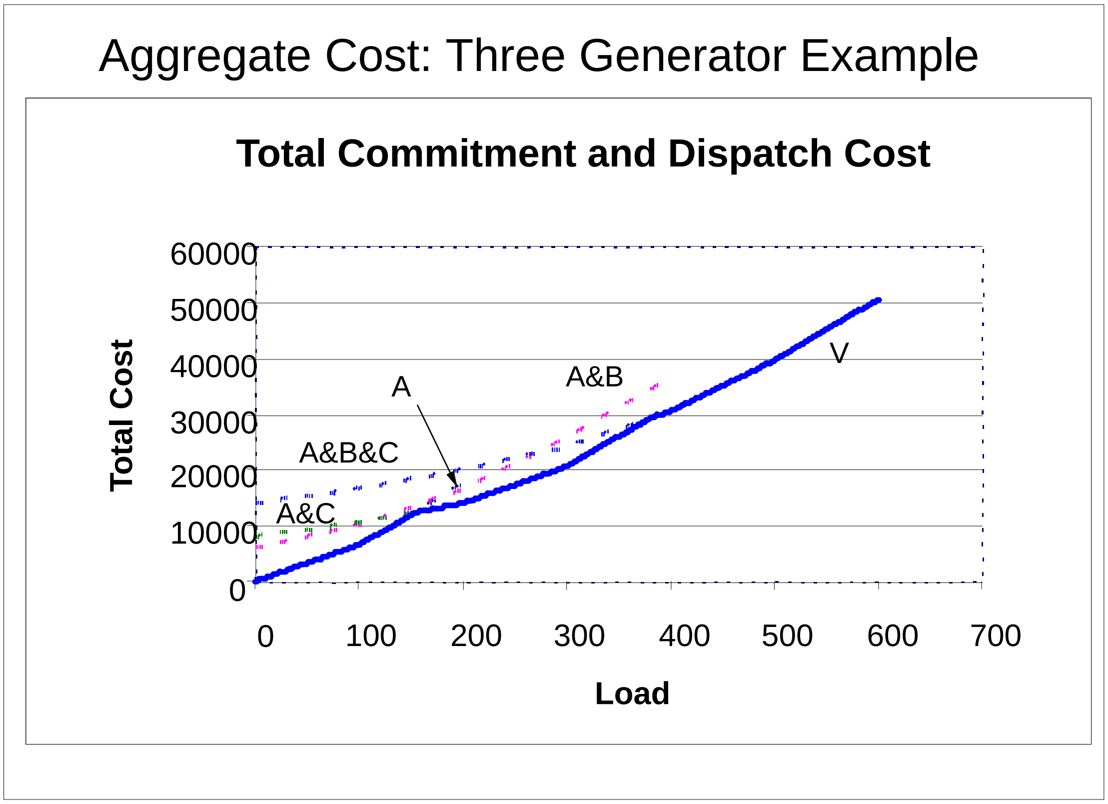


As shown in the figure, aggregate costs follow the pattern of plant A until the load level is high enough (at approximately 178 MW) to support commitment of plant B and switching to the total cost curve of the combination of A & B. Note that in this case the rate of increase of total cost drops, technically, the marginal cost is not monotonically increasing. This change in the marginal cost of meeting an increment of new load is seen more readily in the companion figure showing the implied marginal cost.

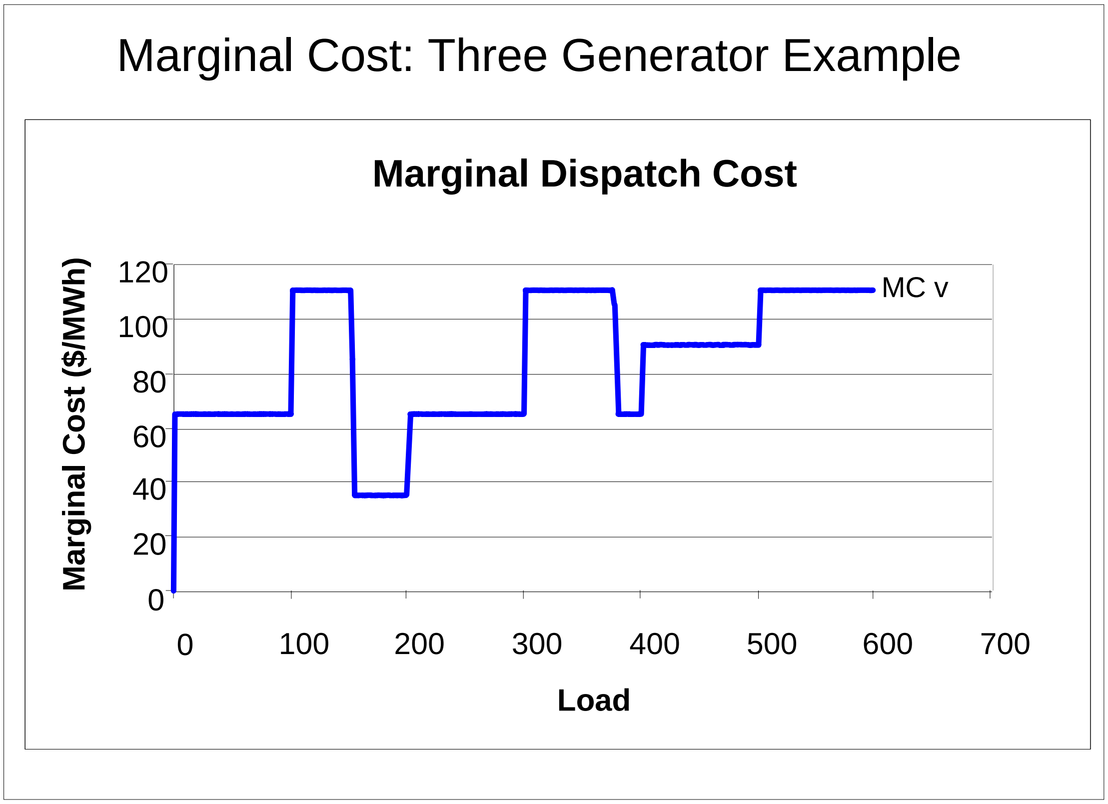


This looks quite different than the well-behaved marginal cost or supply curve in the core model. Now the marginal cost increases and then decreases, and then increases with increasing load. Furthermore, there may be no set of prices that satisfy the market equilibrium conditions that there is "no arbitrage", meaning that suppliers would not want to change the dispatch at the given prices. This raises question of how to define the "market clearing" prices and how to treat other payments needed to support the solution.

The anomalies persist as we consider additional plants and more complicated situations. For example, repeating the analysis with all three plants (A, B, & C) in the illustration produces the following representation of the least-cost value function across different load levels.



The corresponding implied marginal cost curve for this three plant illustration exhibits repeated instances where the commitment decision changes the progression of the implied marginal costs.



As is well known, the discrete commitment variables greatly complicate solution of the economic commitment and dispatch problem. With many plants and many levels of operation, there are too many possible combinations. The same complexity arises in the analysis of the appropriate prices. The further examples that follow emphasize this point for the case of two plants, each with two levels of variable costs. This case is sufficient to illustrate the basic theoretical points in a manner that captures the essential features but is still easy to check.

The task is to develop further the concept of the market-clearing prices and to deal with the additional measures needed to address the key feature that there may be no market clearing prices that support the least-cost solution of the value function $v(y)$.

Energy Prices and Uplift

In the core model for electricity markets, energy prices derived from marginal costs support the equilibrium solution. This is true in the limited sense that within the formal structure included in the model the energy prices provide the appropriate charges to loads and payments to suppliers. However, prices that support the equilibrium solution for energy do not provide the necessary payments for products and services not included in the core model. Ancillary serves such as reactive support, black start capability and so on are not priced in the same way and must be paid for in a matter separate from the formal structure embedded in the core model. The particular rules for determining these payments are often ad hoc and not derived from an inclusive model. Charges applied to customers to cover these costs are similarly based on reasonable but ad hoc rules that often approximate some pro rata allocation across customers. These charges applied in addition to energy payments go under the heading of the "uplift" following a nomenclature established in the UK electricity market restructuring.

Hence, an uplift payment is an inherent part of energy markets. The total cost of uplift payments is usually relatively small and the effects on market incentives are often ignored in formal analysis as being de minimis. However, this may not be true as more and more charges are included in the uplift. When we move to the more general energy model with unit commitment costs and discrete decisions, new opportunities or requirements arise to add to uplift charges. In the more general model, energy prices based on marginal costs will not always be able to support the equilibrium solution. Going further, there may be no set of energy prices that would support an equilibrium solution and something else is required.

One approach that has been suggested, but not applied, is to develop alternative pricing models that might eliminate a need for energy related uplift payments. Since the problem begins with an existence problem—there is no set of prices that would work—these alternative approaches involve some form of pricing rule that replaces the uniform or linear market-clearing prices of the electricity market. A linear price is a single price that is applies to all transactions and the total revenue is simply the price times the quantity, *pq*. A nonlinear pricing rule is anything else that might involve discrimination across transactions or volumes with a rule needed to determine the total revenue.[3] Such rules might violate usual prescriptions for non-discrimination through uniform energy prices. In addition, the rules could create added incentive problems that would raise other difficulties. Although this is an area of possible research, it is not pursued further in the present discussion which addresses linear or uniform energy pricing models.

With linear or uniform market clearing prices, the need arises for uplift payments.

> "Absent a nonlinear pricing scheme, the potential for confiscation could lead generators to withhold themselves from the market or to distort the cost or constraint parameters in their offers to ensure themselves sufficiently high energy rents, with the potential to lead to an inefficient commitment. Most ISOs overcome this confiscation problem by paying uniform hourly energy and ancillary service prices with supplemental make-whole payments, which guarantee that a unit will recover any

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[3] Marcelino Madrigal and Victor H. Quintana, "Existence and Uniqueness of Competitive Equilibrium in Units Commitment Power Pool Auctions: Price Setting and Scheduling Alternatives," IEEE Transactions on Power Systems, Vol. X, No. X, October 2001, pp. 100-108. Marcelino Madrigal, Victor H. Quintana and Jose Aguado, "Stable Extended–Pricing to Deal with Multiple Solutions in Unit Commitment Power Pool Auctions, IEEE Porto Power Tech 2001 Conference Proceedings, September 2001. Alexis L. Motto and Francisco D. Galiano, "Equilibrium of Auction Markets with Unit Commitment: the Need for Augmented Pricing," IEEE Transactions on Power Systems, Vol. 17, No. 3, August 2002, pp. 798-805. Shangyou Hao and Fulin Zhuang, "New Models for Integrated Short-Term Forward Electricity Markets", IEEE Transactions on Power Systems, Vol. 18, No. 2, May 2003, pp. 478-485. Jeovani E. Santiago Lopez and Marcelino Madrigal, "Equilibrium Prices in Transmission Constrained Electrcity Markets: Non-Linear Pricing and Congestion Rents," November 1, 2003.

> portion of its offer-based costs not covered by inframarginal energy and ancillary service rents over the planning horizon."[4]

The "make-whole payments" are part of the aggregate uplift charges. In the case of energy and unit commitment costs, the need for uplift payments can arise because the generator has an incentive to change the commitment or dispatch. For example, given the uniform energy price applicable to a particular plant, the unit commitment and economic dispatch solution may not produce enough energy revenue to cover the total fixed and variable costs. The deficit should be bounded by the total fixed costs, but with only the energy payments the generator would be operating at a loss and would prefer not to be committed. The uplift payment makes the generator whole and adds the needed support for the dispatch.

Another possibility is that a generator is partially dispatched and has remaining unused capacity. If the energy price is above its variable cost, the profit maximizing solution might be to increase output and upset the aggregate energy balance. This condition cannot occur in the core model, but it can arise in the more general framework. Depending on how the energy price is determined the generator sees opportunity costs in foregone profits from complying with the dispatch. An uplift payment for the opportunity cost makes the generator whole and further supports the dispatch.

A more extensive form of opportunity costs arises for the case of uncommitted plants that would be profitable at the uniform energy prices. Again, this condition does not arise in the core model but may well exist for any given set of energy prices in the more general framework. While it is somewhat more controversial to compensate generators who are constrained off, this is another form of opportunity cost and these payments have been part of the uplift charges that support the economic dispatch.

These various forms of energy and commitment cost uplift charges reduce to a simple general principle and calculation. Given the energy price, we can calculate the energy profit or loss that would be earned by each generator at the proposed equilibrium solution. Given the same energy price, we can calculate the profit maximizing position for that same generator. In the core model, the two results would be the same. In the more general model there could be a difference and this difference reveals the make-whole payment that supports the proposed commitment and dispatch decision.

The analysis below formalizes this view of the necessary uplift payments. The focus is on the unit commitment and energy dispatch with uniform energy prices. The uplift is treated as separate from the formal model with de minimis incentive effects. In assuming that the uplift is small and has small incentive effects, we do not consider the rules for allocation of the uplift. However, the analysis of different energy pricing rules addresses the implications for associated uplift payments and provides a basis for evaluating alternative uplift magnitudes.

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4 Ramteen Sioshansi, Richard O'Neill, and Shmuel S. Oren, "Economic Consequences of Alternative Solution Methods for Centralized Unit Commitment in Day-Ahead Electricity Markets," January 2007. (http://www.ieor.berkeley.edu/~ramteen/papers/mip_lr.pdf).

Since there is no unique definition of the energy price, there are alternative price models and associated uplift costs.

Alternative Price Models

The discussion of the need for uplift payments recognizes that the extensions to the core dispatch model are important in defining market equilibrium prices. In the core model, with continuous variables and well-behaved cost functions, the analysis points clearly to the marginal costs as the appropriate prices.

In the more general case with the discrete variables of the unit commitment decisions, there is not an obvious set of prices to use. If there is to be a market-clearing price, uplift payments will be necessary to support the solution. And different definitions of the appropriate market-clearing price have different implications for the nature of the uplift payments.

To address alternative possible pricing conventions, redefine the general problem notation slightly to distinguish the discrete variables. Here we assume that part of feasible set X is the requirement that some of the variables take on the values zero or one, these variables may represent the on/off status of a generating unit. Restate the original problem with explicit identification of the integer constraints.

$$\begin{aligned} v(y) = \underset{(x,u)\in X}{Min} \quad & f(x,u) \\ s.t. \quad & g(x) = y \\ & u = 0,1. \end{aligned}$$

In terms of this restated general problem, the constraints induce prices. We address three alternatives to define market clearing energy prices: a restricted model that comes closest to the strict definition of marginal cost; a dispatchable model that relaxes the discrete requirement and assumes that all units are fully dispatchable and pro-rates the fixed costs; and a model that uses the convex hull of the value function to find the best well-behaved convex approximation that emulates the properties of the core model.

Restricted Model

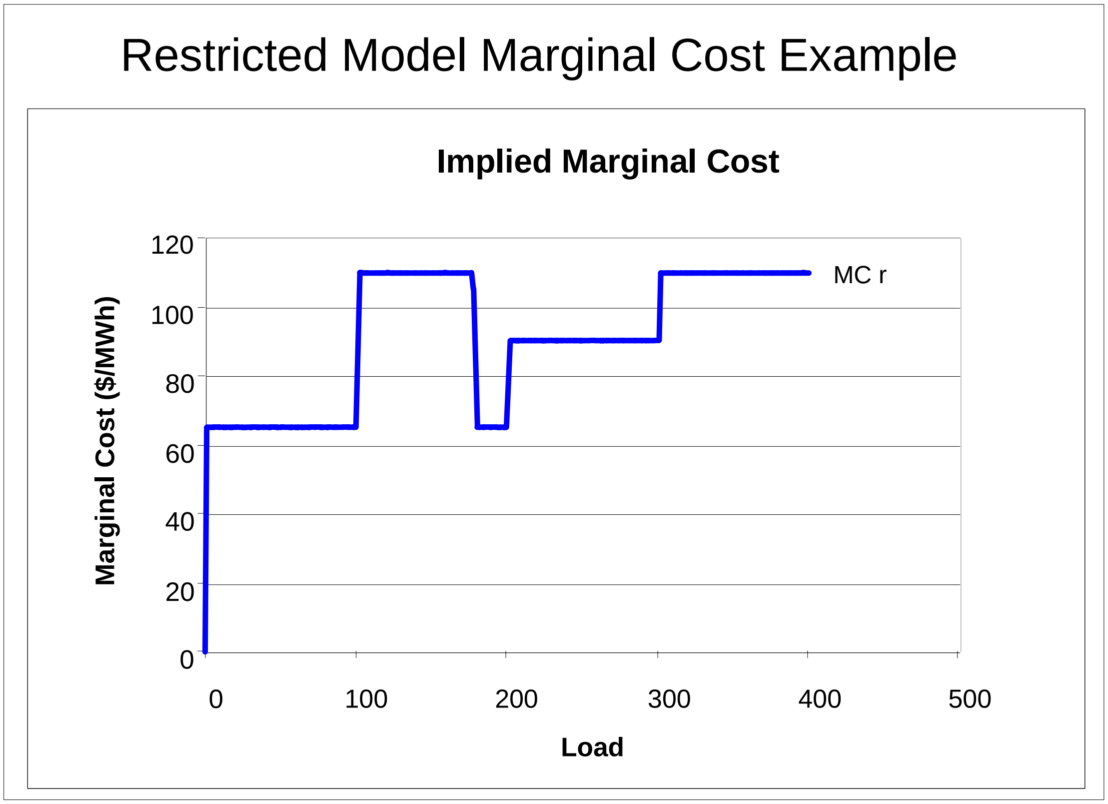
The first approximation provides a pricing definition and interpretation conditioned on knowing the optimal commitment decisions. Given the optimal solution x^O, u^O, restrict (“r”) the model to match the optimal commitments as in:

$$\begin{aligned} v^r(y) = \underset{(x,u)\in X}{Min} \quad & f(x,u) \\ s.t. \quad & g(x) = y \\ & u = u^O. \end{aligned}$$

This restricted model is proposed in O'Neill et al.[5] If the underlying problem is convex for other than the integer requirements, then this restricted problem is a well-behaved problem that yields market clearing prices for both energy load in y (associated with the constraints in $g(x)$) and for capacity commitment (associated with the constraints $u = u^O$).

An attractive feature of this formulation is that with the restriction the model reduces to a standard convex optimization problem subject to the usual range of analyses of the prices and associated properties. In effect, the approach embeds the problem in a higher dimension including the pricing of commitment variables (*u*) as well as energy load. Then the restriction limits the application to the solutions which match the optimal commitment. The solution of the restricted model reproduces the economic dispatch and provides uniform energy prices.

Over the range where $v^r(y)$ is differentiable, the restricted model produces implied marginal costs exactly equal to the marginal costs described above. Hence, in the case of the two plant example, the restricted model implied marginal costs appear as:



[5] Richard P. O'Neill, Paul M. Sotkiewicz, Benjamin F. Hobbs, Michael H. Rothkopf, William R. Stewart, Jr., "Efficient Market-Clearing Prices in Markets with Nonconvexities," European Journal of Operational Research, vol. 164, pp. 269–285.

The marginal costs may not support the equilibrium solution, but with the appropriate definition of uplift payments to compensate for deviations from the equilibrium, these marginal costs could be used as the market prices.

The prices associated with the restriction constraints (i.e., the requirement that u equal the optimal commitment u^O) could be viewed as the components of an uplift calculation. In the derivation of the standard results from the optimization model, the prices for the commitment decisions can be both positive and negative. In the fully linear model, the effect of the commitment prices is to capture all the scarcity rents and leave the short-run profit for each generator exactly zero. The approach suggested in O'Neill et al. is to calculate all the commitment prices but to apply only those that are positive. In effect, this would leave the scarcity rents with generators who earn them at the market clearing price, and pay generators the uplift needed when the market prices do not cover all their costs. The analysis below addresses these uplift payments in the comparison of the implications of the alternative pricing models.

Dispatchable Model

The second model often discussed follows a procedure motivated by the treatment of inflexible units by the New York Independent System Operator as part of the New York electricity market design.[6] The basic idea is to approximate the aggregate cost function with a closely related well-behaved optimization model with continuous variables. In essence, the idea is to replace the integer requirement with a simple set of bounds but within those bounds treat the commitment variables as continuous:

$$\begin{aligned} v^d(y) = \underset{(x,u)\in X}{Min} \quad & f(x,u) \\ s.t. \quad & g(x) = y \\ & 0 \le u \le 1. \end{aligned}$$

In the context of the unit commitment problem, this amounts to treating all the plants as continuously dispatchable ("d") with modified variable costs that include a pro rata share of the fixed costs averaged across the full capacity of the plant. This too yields a well-behaved optimization problem and produces marginal costs and proposed market clearing energy prices associated with the constraints in $g(x)$. An attractive feature of this model is its simplicity. It would be easy to implement using the standard economic

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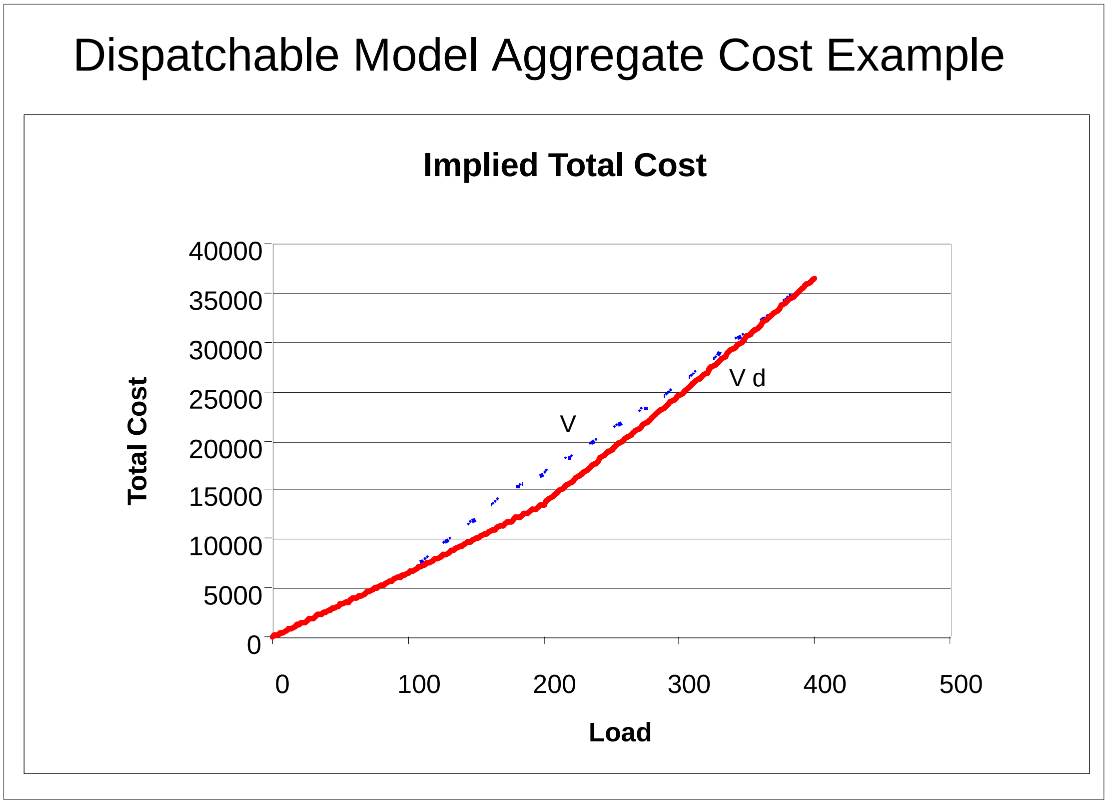
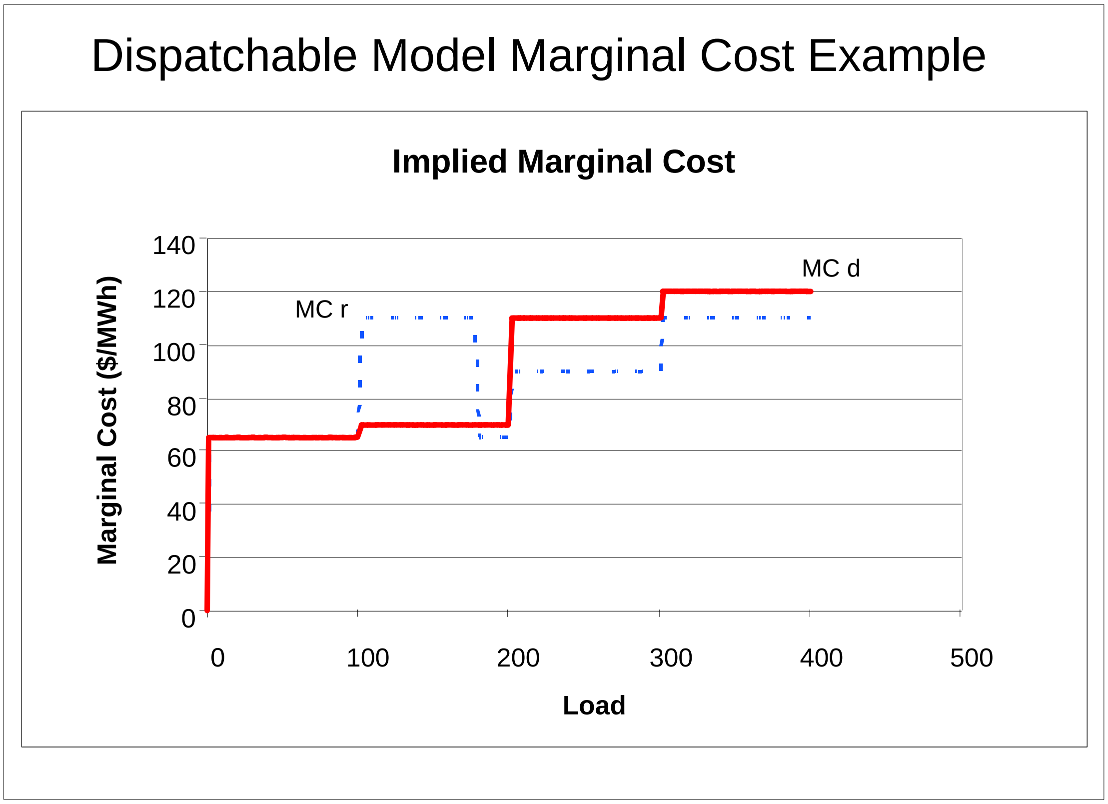
6 "Real-time prices are set by the ideal dispatch pass, in which inflexible (i.e., they must operate at zero or their maximum output) gas turbines are dispatched economically over their entire operating range, even if they are not actually capable of running at anything other than zero or their maximum output." Federal Energy Regulatory Commission Order, FERC Docket No. ER05-1123-000, July 19, 2005. New York Independent System Operator, Inc. FERC Electric Tariff, http://www.nyiso.com/public/webdocs/documents/tariffs/market_services/att_b.pdf "...Fixed Block Units, Import offers, Export Bids, virtual supply and demand Bids and committed non-Fixed Block Units are dispatched to meet Bid Load with Fixed Block Units treated as dispatchable on a flexible basis. LBMPs are calculated from this dispatch." Third Revised Sheet No. 331.01.07. See also http://www.nyiso.com/public/webdocs/documents/tariffs/market_services/att_c.pdf for a description of the uplift payments.

dispatch models by simply treating the units with fixed costs as being committed and dispatchable with a modified variable cost. Solution of the model would not replicate the economic dispatch, but it would produce an implied uniform energy price.

Using the assumptions of the two plant example, we can illustrate the aggregate cost approximation of the total commitment and dispatch costs in the dispatchable model. By construction, the dispatchable value function always lies at or below the aggregate cost function ($v^d(y) \le v(y)$).



The implied aggregate cost of the dispatchable model follows a form similar to the result of the core model. This yields a pattern of marginal costs that are increasing in load and look like a standard supply curve. The implied marginal costs are both below and above the restricted model marginal costs.



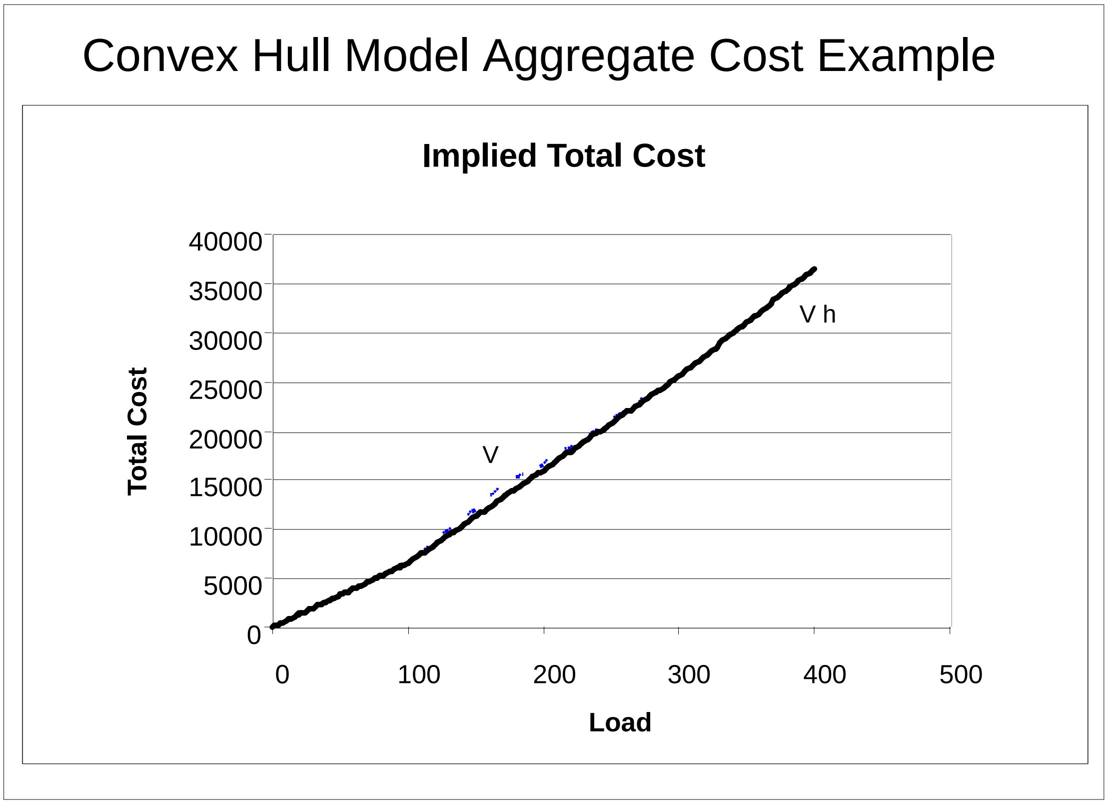
Although the marginal cost curve is more like a conventional supply curve, the implied prices by themselves may not support the market equilibrium solution. There are times when these dispatchable prices yield outcomes where generators would prefer to produce more or less than the equilibrium solution, and uplift payments will still be required.

Convex Hull Model

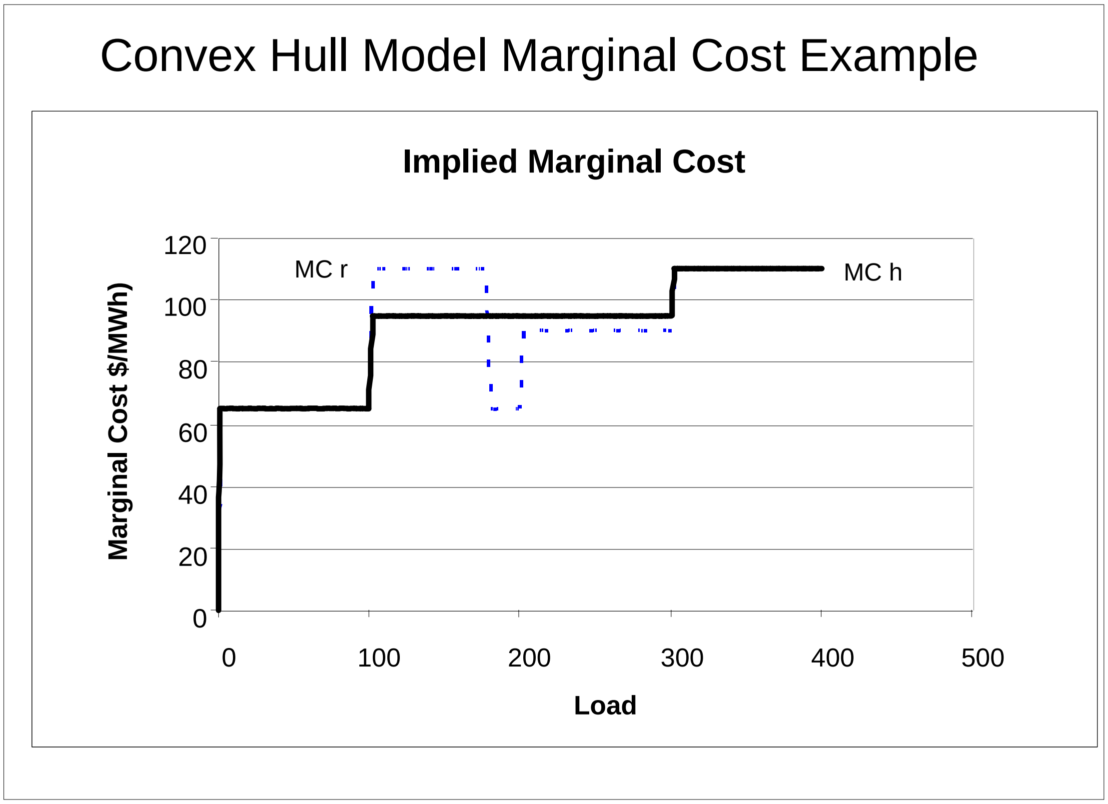
One way to frame the third alternative pricing model is as an alternative well-behaved convex approximation of the aggregate cost function. Any convex function will produce a supply function that has marginal costs increasing in load. The dispatchable model is an example of a convex function that provides a lower bound for the aggregate cost function.

The convex hull of a function is the convex function that is the closest to approximating the function from below. In other words, the convex hull $v^h(y)$ of $v(y)$ is the greatest convex function that is also everywhere such that $v^h(y) \le v(y)$. Set aside for the moment how to obtain the convex hull in general. We can examine the implications of this function for approximation of the aggregate total costs and derivation of the associated marginal cost curve.

Using the two plant example, the convex hull yields total cost as in:



By construction, the total cost of the convex hull is as close as possible to the total aggregate cost in $v(y)$ while preserving the well-behaved properties of the core model. The convex hull approximation will not reproduce the economic dispatch, but it will provide increasing uniform energy prices. The resulting implied marginal cost or supply curve appears as:



The marginal cost is increasing in load. However, for the same reasons as for the other approximations, use of these marginal costs may produce energy prices that would not support the equilibrium solution. There would still be a need for a determination of uplift payments to compensate for the lack of any energy price that by itself could support the outcome.

Uplift Payments

The definition of uplift payment applied here begins with the proposed market clearing price, p. The revenues received for meeting load y will be py and the cost of meeting these loads will be $v(y)$. Hence, the profit or loss of the preferred solution would be

$$\pi(p,y) = py - v(y).$$

Faced with prices p, competitive generators would seek to offer supply z that maximize profits by solving the problem

$$\pi^*(p) = \underset{z}{Max}\{pz - v(z)\}.$$

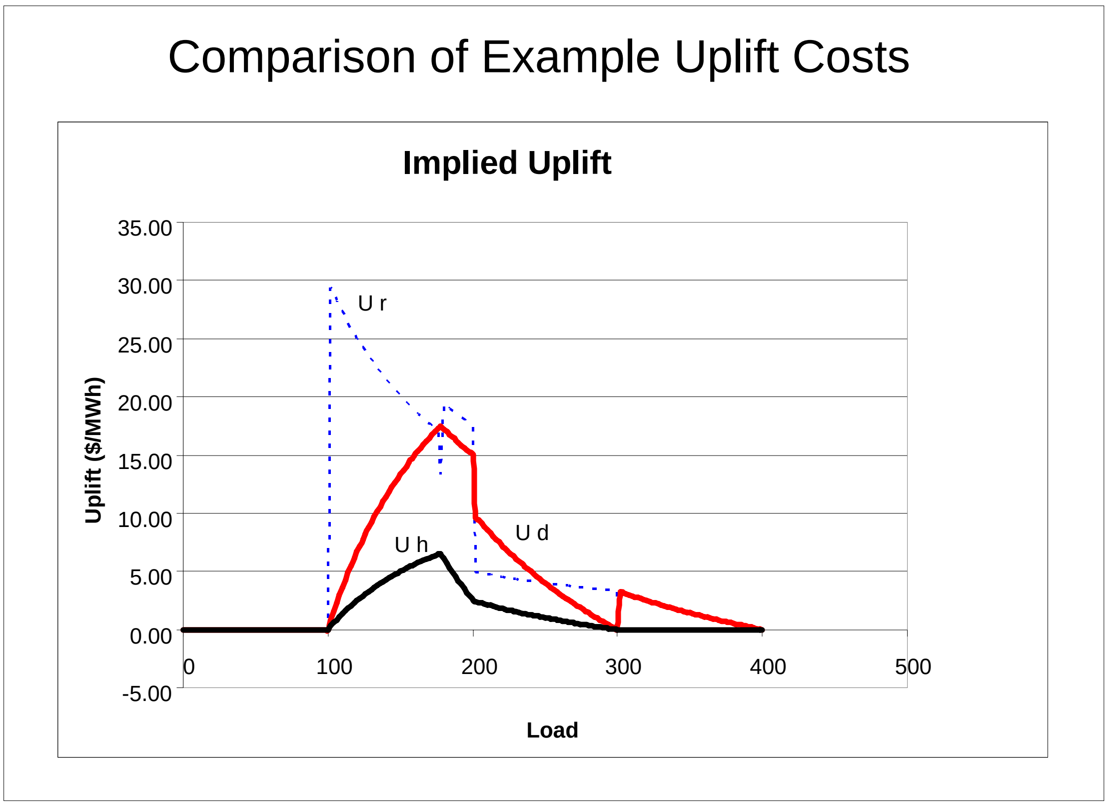
Suppose that this unconstrained profit solution includes y as an optimal solution. Then we say that p supports the equilibrium solution y. This is always the case for the core model of electricity markets, and will be true for many unit commitment and dispatch solutions which could be supported by a market-clearing price.

However, when y is not a profit-maximizing solution in this sense, the price p does not support the solution. Then in order to support the solution the uplift payment would have to make the market participants indifferent between the proposed solution and the unconstrained profit. Hence, the definition of the uplift used here is the difference between the actual energy profits at the proposed solution and the optimal profits given the proposed price:

$$Uplift(p, y) = \pi^*(p) - \pi(p, y).$$

Under this rubric, the usual market payments of the core model are py and produce profit $\pi(p, y)$. The uplift payment compensates for losses or foregone opportunity costs to make the suppliers whole when they accept the proposed solution, as long as they receive the uplift payment in addition to the direct payments in the energy market at prices p.

Applying this definition to the three models above we calculate the uplift payments per megawatt for the two plant example and compare the three cases in the figure:



Below 100 MW the approximating functions are identical to the aggregate cost function and there are no uplift payments. In the intermediate range the uplift payments for the restricted model and its volatile marginal cost curve produce high uplift payments, sometimes much greater than the uplift payments of the dispatchable model. Above 300

MW, the restricted model and the convex hull model are identical to the aggregate cost function and there are no uplift payments. However, above 300 MW there continue to be uplift payments in the dispatchable model because the implied energy prices are above the highest variable cost segment.

Importantly the uplift payments for the prices associated with the convex hull approximation are always less than the uplift payments for the other two alternatives. Although the uplift payments for the other two pricing models are sometimes higher and sometimes lower, as shown below the convex hull model always has the lowest uplift payments. Therefore, marginal cost prices associated with the convex hull are the minimum uplift prices proposed by Ring.[7] This relationship is not a coincidence and does not depend on the particular assumptions of the example.

The convex hull approximation always results in energy prices that are increasing in load. As discussed further below, the prices from the convex hull approximation also produce the minimum possible uplift of any set of uniform energy prices. Furthermore, the convex hull prices are the same as the dual solution for a natural formulation of the dual problem. This connection to duality theory is important both for conceptual reasons and in guiding us towards feasible computational approaches for obtaining the appropriate prices.

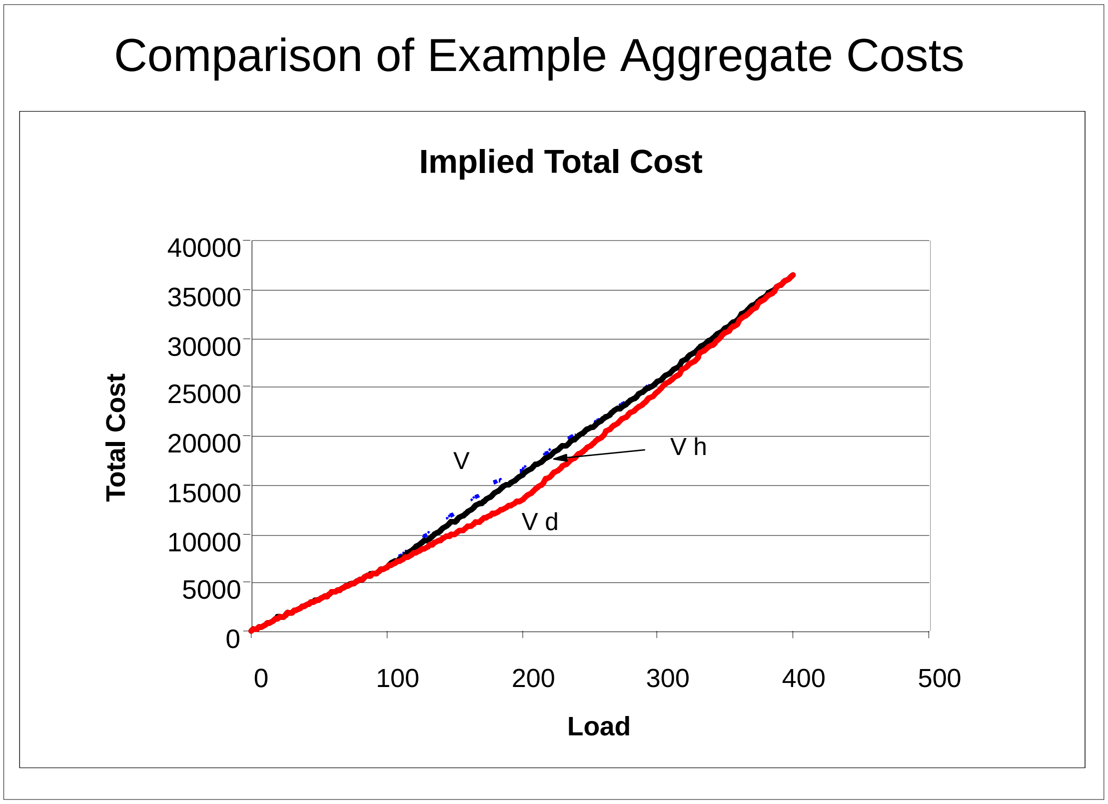
Price Comparisons

For well-behaved problems that are otherwise convex except for the integer constraints, these price model approximations are convex in *y* and we can solve for the corresponding price support. For these alternative price models, we can have values and supports such that:

$$v^d(y) < v^h(y) < v^r(y) = v(y),$$
$$p^d \neq p^h \neq p^r.$$

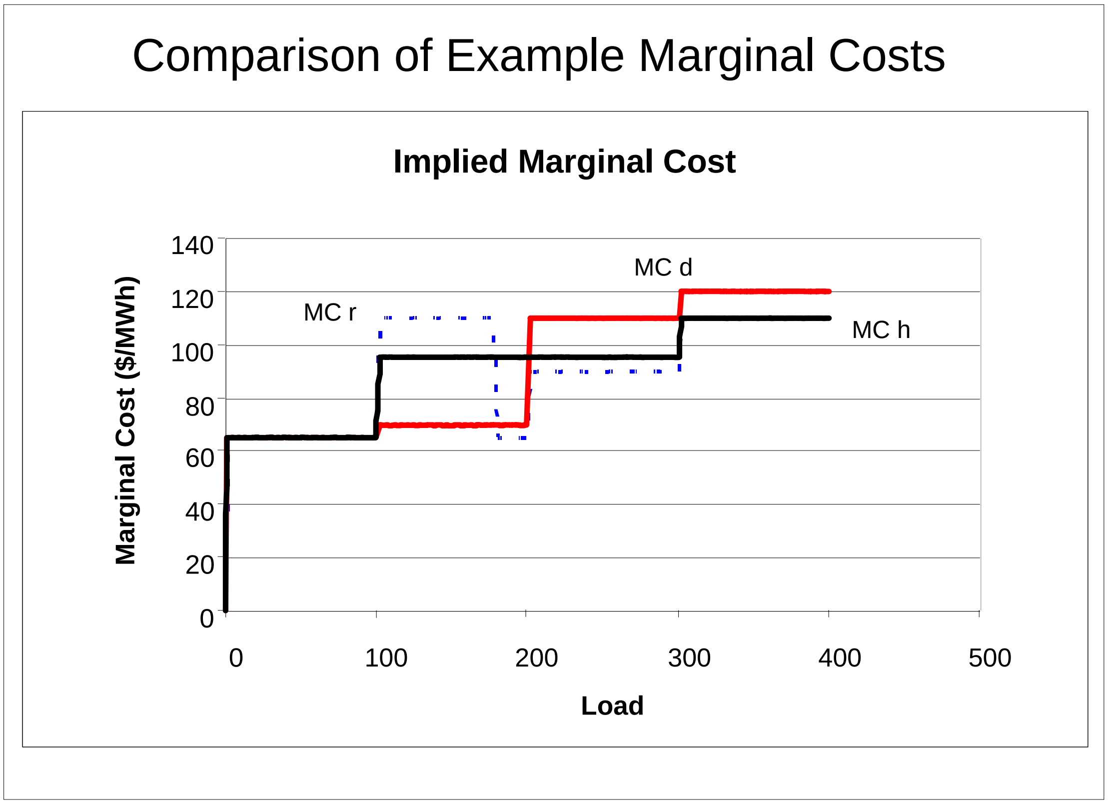
Comparing the approximate costs for the illustrative two plant example, we have:

[7] Brendan J. Ring, "Dispatch Based Pricing in Decentralized Power Systems," Ph.D. thesis, Department of Management, University of Canterbury, Christchurch, New Zealand, 1995. (see the HEPG web page at http://ksgwww.harvard.edu/hepg/). Ring emphasized the linear case and focused on the uplift payments, with the idea of minimizing these payments as a "best compromise" to provide workable pricing methods in the presence of deviations from the idealized model.

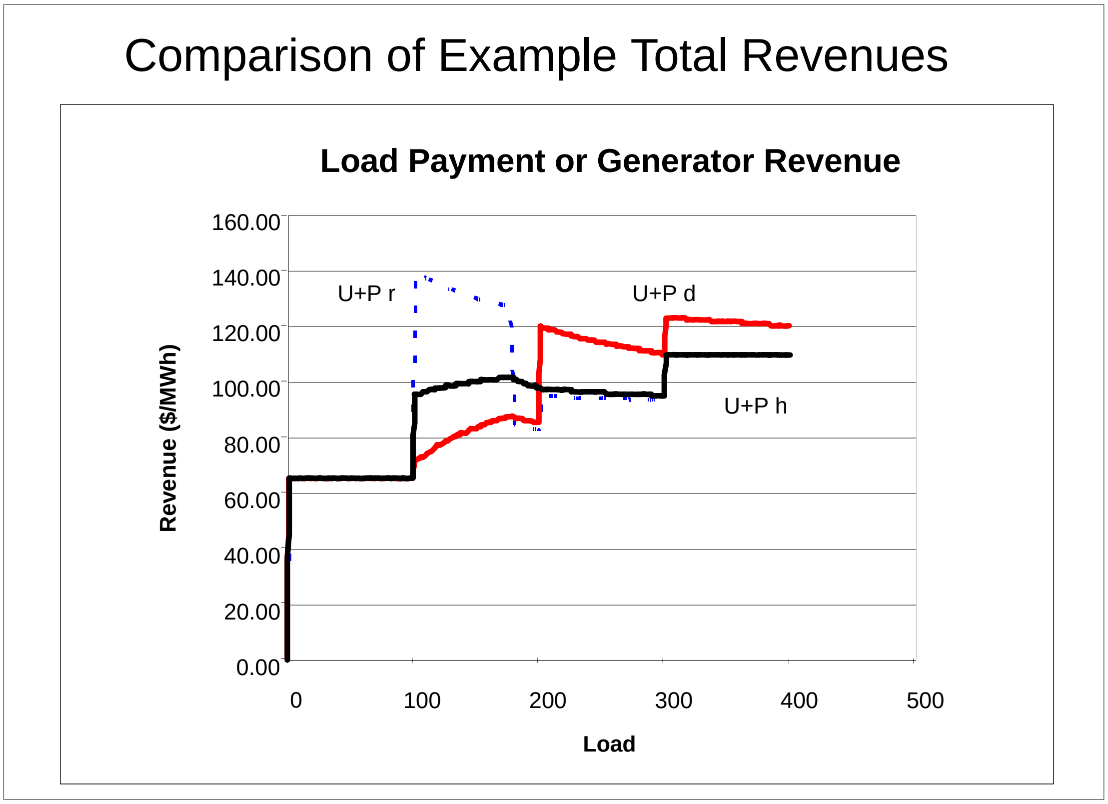


The restricted pricing problem is easy to solve, but it produces a volatile price and uplift combination.[8] The relaxed problem is also easy to solve, but it produces a price that may result in a large uplift. The convex hull problem requires a method of characterization and solution but assures the minimum uplift. The dispatchable and convex hull models produce an increasing energy supply curve.

[8] William W. Hogan and Brendan R. Ring, "On Minimum-Uplift Pricing for Electricity Markets," March 19, 2003, (available at http://ksghome.harvard.edu/~WHogan/minuplift_031903.pdf).



The combination of energy payments at the uniform energy price and the uplift payment produces the total payments by load and the corresponding revenues to generators.



Although the energy supply curves are increasing in load for the dispatchable and convex hull models, the uplift is not monotonic and the total revenues increase and decrease with total load. The convex hull model minimizes this impact because it minimizes the uplift.

An examination of the connection with duality theory provides additional clarification and suggests computational approaches for solving the convex-hull, minimum-uplift, pricing problem.

Duality and Minimum Uplift

With this motivation, we formulate the dual of the optimization problem with respect to the complicating constraints and draw the connections to the convex hull approximation, market-clearing prices and uplift payments.

Introduce the vector of prices (a.k.a., Lagrange multipliers) p and the Lagrangian function:

$$L(y,x,p) = f(x) + p\left(y - g(x)\right).$$

The Lagrangian "prices out" the complicating constraints and, given p, produces a problem that is easier to solve. For given prices, define the optimized Lagrangian value as:

$$\hat{L}(y,p) = \underset{x\in X}{Inf}\left\{ f(x) + p\left(y - g(x)\right)\right\}.$$

Note that this definition makes no assumptions about the feasible set X, which may include limitations to discrete choices. The associated dual problem is defined as choosing the prices p^D to maximize the optimized Lagrangian to obtain

$$L^*(y) = \underset{p}{Sup}\,\hat{L}(y,p) = \underset{p}{Sup}\left\{\underset{x\in X}{Inf}\left\{ f(x) + p\left(y - g(x)\right)\right\}\right\}.$$

In the case of a well-behaved convex optimization problem, where the decision variables are continuous and the constraints sets are all convex, the optimal dual solution produces a vector of prices that supports the optimal solution. In particular, using these prices, the corresponding solution for x embedded in $v(y)$ also solves the problem in $\hat{L}(y,p)$. Furthermore, under these conditions, we have

$$L^*(y) = v(y).$$

In the more general situation without the convenient convexity assumptions, there may be no equilibrium prices that support the solution at y and we have a duality gap, where

$$L^*(y) < v(y).$$

To make the connection with the minimum uplift, consider this formulation of the dual problem. For convenience here, the integer constraints are enforced as part of the constraints implicit in the set X and are not represented separately. By definition, for the dual solution we have:

$$\begin{aligned}
L^*(y) &= \underset{p}{Sup}\left\{\underset{x\in X}{Inf}\left\{ f(x) + p\left(y - g(x)\right)\right\}\right\} \\
&\le \underset{p}{Sup}\left\{\underset{x\in X}{Inf}\left\{ f(x) + p\left(y - g(x)\right) \middle| g(x) = y\right\}\right\} \\
&= \underset{p}{Sup}\left\{\underset{x\in X}{Inf}\left\{ f(x) \middle| g(x) = y\right\}\right\} = \underset{x\in X}{Inf}\left\{ f(x) \middle| g(x) = y\right\} = v(y).
\end{aligned}$$

So $L^*(y) \le v(y)$.

The difference $v(y) - L^*(y)$ is known as the duality gap. When equality holds, as when $v(y)$ is convex and certain regularity conditions hold, there is no duality gap.[9]

Gribik presents an alternative representation of the argument:[10]

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[9] Mokhtar S. Bazaraa, Hanif D. Sherali, and C.M. Shetty, Nonlinear Programming: Theory and Algorithms, John Wiley & Sons, 2nd. Edition, 1993, pp. 162-163.

[10] Paul Gribik, Notes (mimeo), July 2007. See also, James E. Falk, "Lagrange Multipliers and Nonconvex Programs," SIAM Journal on Control, Vol. 7, No. 4, November 1969; Dimitri P. Betsekis,

$$\begin{aligned}
\hat{L}(y,p) &= \underset{x\in X}{Inf}\left\{f(x)+p\left(y-g(x)\right)\right\} \\
&= \underset{x\in X,z}{Inf}\left\{f(x)+p(y-z)\middle|g(x)=z\right\} \\
&= py+\underset{x\in X,z}{Inf}\left\{f(x)-pz\middle|g(x)=z\right\} \\
&= py+\underset{z}{Inf}\left\{-pz+\underset{x\in X}{Inf}\left\{f(x)\middle|g(x)=z\right\}\right\} \\
&= py+\underset{z}{Inf}\left\{-pz+v(z)\right\} \\
&= py-\underset{z}{Sup}\left\{pz-v(z)\right\}.
\end{aligned}$$

The Fenchel convex conjugate of v is by definition:

$$v^c(p)=\underset{z}{Sup}\left\{pz-v(z)\right\}.$$

Note that $v^c(p)$ is the supremum over a set of convex (affine) functions of p and is therefore a convex function of p.

Hence,

$$\hat{L}(y,p)=py-v^c(p).$$

Now

$$L^*(y)=\underset{p}{Sup}\,\hat{L}(y,p)=\underset{p}{Sup}\left\{py-v^c(p)\right\}.$$

Therefore, applying the conjugate definition again, we have

$$L^*(y)=v^{cc}(y).$$

The resulting function $v^{cc}(y)$ is a closed convex function of y and we have[11]

Constrined Optimization and Lagrange Multiplier Methods, Athena Scientific, Belmont, MA, 1996, pp. 315-318.

[11] For a discussion of the connection between the convex hull and the dual problem, with an application to the special case of separable problems, see D. Li, J. Wang, and X.L. Sun, "Computing Exact Solution to Nonlinear Integer Programming: Convergent Lagrangian and Objective Cut Method," Journal of Global Optimization, Vol. 39, 2007, pp. 127-154. C. Lemarchechal and A. Renaud, "A Geometric Study of Duality Gaps, with Applications," Mathematical Programming, Series A 90, 2001, pp. 399-427. David E. Bell and Jeremy F. Shapiro, "A Convergent Duality Theory for Integer Programming," Operations Research, Vol. 25, No. 3, May-June 1977, pp. 419-434. Harvey J. Greenberg, "Bounding Nonconvex Programs by Conjugates," Operations Research,, Vol. 21, No. 1, 1973, pp. 346-348. Fred Glover, "Surrogate Constraint Duality in Mathematical Programming," Operations Research, Vol. 23, No. 3, May-June 1975, pp. 434-450. Regarding use of the perturbation function and a penalty function, see D. Li and X.L. Sun, "Towards Strong Duality in Integer Programming," ," Journal of Global Optimization, Vol. 36, 2006, pp. 255-282.

$$v^{cc}(y) = L^*(y) \le v(y).$$

Suppose that $\bar{v}(y)$ is the closed convex hull of $v(y)$. Then[12]

$$v^{cc}(y) = \bar{v}(y) \le v(y).$$

In other words, $v^{cc}(y)$ equals the convex hull of $v(y)$ over *y*. Further, under one of the regularity conditions with p^D as a solution to the dual problem, p^D defines a supporting hyperplane (a.k.a., marginal cost) for $v^{cc}(y)$ at *y*:

$$v^{cc}(z) = \underset{\lambda}{Sup}\left\{\lambda z - v^c(\lambda)\right\} \ge p^D z - v^c\left(p^D\right) = p^D z + v^{cc}(y) - p^D y = v^{cc}(y) + p^D(z-y).$$

Therefore, p^D is a subgradient of $v^{cc}(y) = v^h(y)$,

$$p^D \in \partial v^h(y).$$

In general, the price supports defined by the subgradients are not unique, but all elements of the set characterized by the subdifferential, $\partial v^h(y)$, support the convex hull.

The connection to the uplift depends on a certain economic interpretation of the duality gap. From above

$$\hat{L}(y,p) = py + \underset{z}{Inf}\left\{-pz + \underset{x\in X}{Inf}\left\{f(x) \middle| g(x) = z\right\}\right\}.$$

Hence,

$$\begin{aligned}
L^*(y) &= \underset{p}{Sup}\,\hat{L}(y,p) \\
&= \underset{p}{Sup}\left\{py + \underset{z}{Inf}\left\{-pz + \underset{x\in X}{Inf}\left\{f(x) \middle| g(x) = z\right\}\right\}\right\} \\
&= \underset{p}{Sup}\left\{py + \underset{z}{Inf}\left\{-pz + v(z)\right\}\right\}.
\end{aligned}$$

Therefore, the duality gap is

$$\begin{aligned}
v(y) - L^*(y) &= v(y) - \underset{p}{Sup}\left\{py + \underset{z}{Inf}\left\{-pz + v(z)\right\}\right\} \\
&= v(y) + \underset{p}{Inf}\left\{-py + \underset{z}{Sup}\left\{pz - v(z)\right\}\right\} \\
&= \underset{p}{Inf}\left\{\underset{z}{Sup}\left\{pz - v(z)\right\} - \left[py - v(y)\right]\right\}.
\end{aligned}$$

Given any output z, the economic profit is

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[12] R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, NJ, 1970, p. 104.

$$\pi(p,z) = pz - v(z).$$

This is the difference between the revenues for *z* at prices *p* and the minimum cost of meeting the requirement in *z*. We give an economic interpretation where the first term in the duality gap is the profit maximizing outcome given prices p:

$$\pi^*(p) = \underset{z}{Sup}\{pz - v(z)\} = \underset{z}{Sup}\,\pi(p,z).$$

The actual economic profit without further uplift payments is

$$\pi(p,y) = py - v(y).$$

If we have to make up the difference in order to compensate direct losses or for foregone opportunities, then the total payment is the difference:

$$Uplift(p,y) = \pi^*(p) - \pi(p,y).$$

In other words, the dual problem seeks a p^D that minimizes the uplift.[13] With this interpretation, the duality gap equals the minimum uplift across all possible prices *p*:

$$\begin{aligned} v(y) - L^*(y) &= \underset{p}{Inf}\left\{\underset{z}{Sup}\{pz - v(z)\} - [py - v(y)]\right\} \\ &= \underset{p}{Inf}\{\pi^*(p) - \pi(p,y)\} = \underset{p}{Inf}\, Uplift(p,y). \end{aligned}$$

If p^D is a dual solution and

$$y \in \arg\max_z \{p^D z - v(z)\},$$

then there is no duality gap and no uplift. In this sense, the prices in p^D support the equilibrium solution if there is no duality gap. However, if

$$y \notin \arg\max_z \{p^D z - v(z)\},$$

then p^D does not "support" y, and there is a duality gap equal to the minimum uplift.

Apparently the argument applies to an arbitrary feasible solution with $x^a \in X, g(x^a) = y$. Then

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13 Brendan J. Ring, "Dispatch Based Pricing in Decentralized Power Systems," Ph.D. thesis, Department of Management, University of Canterbury, Christchurch, New Zealand, 1995. (see the HEPG web page at http://ksgwww.harvard.edu/hepg/) proposed choosing prices to minimizing the uplift. Marcelino Madrigal, "Optimization Models and techniques for Implementation and Pricing of Electricity Markets," Ph. D. thesis, University of Waterloo, Canada, 2000, p. 47, describes the connection with duality theory and established an upper bound on the uplift.

$$f\left(x^a\right)-L^*\left(y\right)=\underset{p}{Inf}\left\{\underset{z}{Sup}\left\{pz-v\left(z\right)\right\}-\left[py-f\left(x^a\right)\right]\right\}$$
$$=\underset{p}{Inf}\left\{\pi^*\left(p\right)-\left[py-f\left(x^a\right)\right]\right\}=\underset{p}{Inf}\,Uplift\left(p,y\right).$$

Here we interpret the uplift as the difference between the optimal profit and the actual profit in the arbitrary dispatch. The argument can be extended to include conditions where $g\left(x\right)$ defines mixtures of equality and inequality constraints.

Note that under either definition the uplift is nonnegative.

Ring proposes choosing energy prices to minimize the uplift in the case of approximate solutions to dispatch problems. Ring's analysis includes the case of the discrete unit commitment problems. Using examples with each plant having a single variable cost, Ring showed that the uplift minimizing solution was the same as the solution in the dispatchable price model.[14] In the more general case with multiple segments having different variable costs for the same plant, as shown in the examples above, the dispatchable solution and the minimum-uplift, convex-hull, dual-solution prices and associated uplifts can be different. Hogan and Ring compare the minimum uplift prices with those of the restricted model.[15]

Dispatch-Based Pricing Approximations

The electricity market model utilizes the formulation of a security-constrained economic dispatch. This formulation includes many constraints to represent transmission operations and reliability requirements. Computational approaches for solving these models involve a great deal of art and technique in evaluating and managing the solution procedures. Given the solution, the pricing model can exploit the results of the process to greatly simplify the problem and reduce the dimensionality of the model.

Transmission Constraints

Suppose that the transmission constraints defining the feasible set are

$$K\left(x\right)\leq K_{Max}\,.$$

Then we have the value function of the economic dispatch as:

$$v\left(y,K_{Max}\right)=\underset{x\in X}{Inf}\left\{f\left(x\right)\middle|g\left(x\right)=y,K\left(x\right)\leq K_{Max}\right\}.$$

There are many elements to account for all the possible contingency constraints. We could define the set of "binding" constraints given a solution *x* as

[14] Brendan J. Ring, "Dispatch Based Pricing in Decentralized Power Systems," Ph. D. thesis, Department of Management, University of Canterbury, Christchurch, New Zealand, 1995, p. 203.

[15] William W. Hogan and Brendan R. Ring, "On Minimum-Uplift Pricing for Electricity Markets," March 19, 2003, (available at http://ksghome.harvard.edu/~WHogan/minuplift_031903.pdf).

$$\bar{K}(x) = \left\{ K_j(x) \middle| K_j(x) = K_{jMax} \right\}.$$

In a convex case of the core model, we can drop the non-binding constraints and not change the solution.[16] This is not true in the more general formulation with the non-convex unit commitment constraints and variables.

A slightly more general solution would be to find a small set of "limiting" constraints such that the solution does not change. This would be any subset $\tilde{K}(x)$ such that

$$v\left(y, \tilde{K}_{Max}\right) = \underset{x \in X}{Inf} \left\{ f(x) \middle| g(x) = y, \tilde{K}(x) \le \tilde{K}_{Max} \right\}.$$

Given the information in the actual dispatch, x^a, we seek a small subset that drops most of the elements of $K(x)$. If no constraints can be dropped, then this is the full constraint set. Given these constraints we modify the approximation further by linearizing the constraints in $\tilde{K}(x)$.

With the appropriate linearization and dualization with respect to the constraints in $\tilde{K}(x)$ or any other coupling constraints, the model simplifies to be separable across many components (generators) that make up the cost function.

Reliability Commitment

In organized markets with organized unit commitment and dispatch, a reliability concern arises in a potential conflict between equilibrium load solutions and operator load forecasts. In the bid-based day-ahead models, load bids could in principle define the total load to be met and the solution could be obtained consistent with that load. However, system operators also regularly forecast load over the short horizon and seek to maintain reliable conditions to meet that forecast load.

If the two load levels differ, the question arises as whether to solve the unit commitment problem to meet the bid-in load or the operator-forecasted load. Even economists would pause at relying solely on the perfection of the market to address this reliability question. System operators argue strongly that deference must be given to preserving reliability under the forecast load.

The resolution of this issue has been to adopt a heuristic method that follows some variant of a three step procedure. The first step would be to solve for the economic commitment and dispatch using the bid-in load. Then in a second step with the economically committed units forced on, solve a related commitment and dispatch problem with the forecast load. In the second step, the related problem has a reliability focus and uses only the fixed costs of commitment but treats all the variable dispatch costs as zero. The intent is to minimize the incremental costs imposed by the reliability

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[16] Ring, 1995. See also William W. Hogan, E. Grant Read and Brendan J. Ring, "Using Mathematical Programming for Electricity Spot Pricing," International Transactions in Operational Research, IFORS/Elsevier, Vol. 3, No. 3/4 1996, pp. 209-221.

requirement.[17] A third step can then be included to solve for the economic dispatch keeping the combined commitment decisions from the second step.

One way to view this problem would be to see the forecast load as adding a (very large) set of constraints, doubling the size of the commitment and dispatch problem. The sequential three-step procedure is an ad hoc method to meet the reliability requirement while avoiding this currently prohibitive computational task.

In the context of price and uplift determination, the implication of the sequential method is to fix the commitment but not otherwise represent the added reliability constraints in the pricing model. In determining prices and uplift, this is equivalent to picking a feasible but not necessarily optimal solution in the simplified pricing model, as discussed above. The price from the first stage dual problem without the reliability constraint still provides the minimum uplift result.

Computational Methods

Solving the unit commitment and economic dispatch problem involves extensive computations that present serious challenges. Methods for solving these mixed-integer programs have advanced to the point where they are a regular production tool.[18] In some cases, application of a Lagrangian relaxation method might be used in the search for a solution to the unit commitment problem.[19] A concern with these dual methods is that they may not produce a primal feasible solution. However, these methods would produce a dual price as a by-product.

In other cases where a dual price is not available, separate calculations may be required to obtain the appropriate energy prices. One attraction of both the restricted and the dispatchable models is that each offers a straightforward computational model for obtaining the implied energy prices. The convex-hull, minimum-uplift model presents a less obvious solution method. It would always be possible to simply apply the Lagrangian techniques directly, but a more focused approach for dispatched-based pricing would be preferred that exploits information in the proposed commitment and dispatch.

In the case that a dual price vector is not available as a by-product of solving the unit commitment problem, a further characterization of the solution suggests an algorithm for obtaining:

$$p^D = p^h \in \partial v^h\left(y\right).$$

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[17] Michael D. Cadwalader, Scott M. Harvey, William Hogan, and Susan L. Pope, "Reliability, Scheduling Markets, and Electricity Pricing," May 1998, available at (www.whogan.com).

[18] D. Streiffert, R. Philbrick, and A. Ott, "A Mixed Integer Programming Solution for Market Clearing and Reliability Analysis," in Power Engineering Society General Meeting, 2005, IEEE, San Francisco, CA, 2005.

[19] Marshall L. Fisher, "The Lagrangian Relaxation Method for Solving Integer Programming Problems," Management Science, Vol. 27, No. 1, January 1981, pp. 1-18.

In the unit commitment problem, there is a natural separability by unit, assuming that we dualize or price out the joint constraints. Suppose

$$f(x) = \sum_i f_i(x_i)$$

$$g(x) = \sum_i g_i(x_i)$$

$$X = \prod_i X_i$$

$$v(y) = \underset{\{x_i \in X_i\}}{Inf} \left\{ \sum_i f_i(x_i) \middle| \sum_i g_i(x_i) = y \right\}.$$

Then

$$\begin{aligned} v(y) &= \underset{\{x_i \in X_i\}}{Inf} \left\{ \sum_i f_i(x_i) \middle| \sum_i g_i(x_i) = y \right\} \\ &= \underset{\{z_i\}}{Inf} \left\{ \sum_i \left[\underset{x_i \in X_i}{Inf} \left\{ f_i(x_i) \middle| g_i(x_i) = z_i \right\} \right] \middle| \sum_i z_i = y \right\} \\ &= \underset{\{z_i\}}{Inf} \left\{ \sum_i v_i(z_i) \middle| \sum_i z_i = y \right\}. \end{aligned}$$

Following a similar argument, in the separable case we can write the dual problem as

$$\begin{aligned} L^*(y) &= \underset{p}{Sup} \left\{ py - \underset{z}{Sup} \left\{ pz - v(z) \right\} \right\} \\ &= \underset{p}{Sup} \left\{ py - \sum_i \left\{ \underset{z_i}{Sup} \left\{ pz_i - v_i(z_i) \right\} \right\} \right\}. \end{aligned}$$

Using the Fenchel conjugate again, therefore, we have

$$L^*(y) = \underset{p}{Sup} \left\{ py - \sum_i v_i^c(p) \right\}.$$

We know that $v_i^{cc}(z_i) = v_i^h(z_i)$ is the convex hull of $v_i(z_i)$. Furthermore, $v_i^c(p) = v_i^{hc}(p)$. In other words, the conjugate of v_i is also the conjugate of its convex hull.

Apparently,

$$L^*(y) = \underset{p}{Sup} \left\{ py - \sum_i v_i^{hc}(p) \right\}.$$

But this is the same as the dual problem obtained by substituting the convex hulls of the components as in:

$$v^*(y) = \underset{\{z_i\}}{Inf} \left\{ \sum_i v_i^h(z_i) \middle| \sum_i z_i = y \right\}.$$

Hence, the convex hull of v^* is also L^*. Since v^* is itself a convex function, we have

$$L^*(y) = v^h(y) = v^*(y) \le v(y).$$

This provides an alternative characterization of the convex hull of $v(y)$.[20] In many cases, it is easy to describe the convex hull of the individual components. Given these components we could solve the convex optimization problem directly to obtain a dual solution (with no duality gap) that would be the dual prices for the original non-convex problem. This would not necessarily reproduce the economic dispatch but would provide a price for the dual solution.

Even when we cannot write down the full convex hull of each component in a convenient way, the formulation of v^* offers an alternative way to generate supports or cuts in a cutting plane approach to solving the dual problem.

In the unit commitment case, if there is a startup cost, then $v_i(0) = 0$ is a point of discontinuity. Assuming the rest of the cost function is convex away from this point of discontinuity, the convex hull is of a simple form connecting the origin and a point on a convex function. Given a good solution to the primal problem for the vector of dispatch decisions across segments and periods, x^O, then:

1. For each non-convex v_i, let $z_i^O = g_i(x_i^O)$. If $z_i^O = 0$, pick an arbitrary point near zero. Find a point z_i^1 on the ray through the origin and z_i^O where there is a price support for v_i with $v_i(z_i^1) = p_i^1 z_i^1$. Even for a multi-period problem, this is a one dimensional search and is easy to do for the simple dispatch problem of a separable unit. By construction, p_i^1 is also a support for the convex hull, v_i^*. Use p_i^1 and z_i^1 to construct a convex approximation of v_i^*, say $\tilde{v}_i^*$. For the convex cost functions $\tilde{v}_i^* = v_i$.

2. Solve for the dual price in $\tilde{v}^*(y) = \underset{\{z_i\}}{Inf} \left\{ \sum_i \tilde{v}_i^*(z_i) \middle| \sum_i z_i = y \right\}$. Let this be p^1. Let k=1.

3. For each unit, solve $\underset{z_i}{Max}\left\{ p^k z_i - v_i(z_i) \right\}$ for z_i^{k+1}. Let k=k+1.

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[20] See the same result for unit commitment problems with linear constraints in C. Lemarchechal and A. Renaud, "A Geometric Study of Duality Gaps, with Applications," Mathematical Programming, Series A 90, 2001, pp. 419. James E. Falk and Richard M. Soland, "An Algorithm for Separable Nonconvex Programming Problems," Management Science, Vol. 15, No. 9, Theory Series, May, 1969, pp. 550-569.

4. Solve $\underset{p}{Max}\left\{ py - \sum_i \sigma_i \middle| \sigma_i \geq p z_i^k - v_i\left(z_i^k\right), \forall k \right\}$ for p^k. Check for convergence to stop or return to step 3.

This is a heuristic method for getting a good initial solution and support. Starting with step 3 the method becomes a standard outer approximation method.[21] If the underlying problem is piecewise linear, the solution should be obtained in a finite number of steps. A simple example illustrates.

Consider another example with two plants, but simplified to have only one variable cost segment. Both have fixed costs. Plant 1 has a positive variable cost, and plant 2 has zero variable cost. Both plants have limited capacity.

$$
\begin{aligned}
& v(y) = \underset{x_1, x_2, u_1, u_2}{Min} \quad F_1 u_1 + c_1 x_1 + F_2 u_2 \\
& s.t. \\
& 0 \leq x_1 \leq K_1 u_1 \\
& 0 \leq x_2 \leq K_2 u_2 \\
& u_1 = 0,1 \\
& u_2 = 0,1 \\
& x_1 + x_2 = y.
\end{aligned}
$$

The convex hulls for the individual plants utilize the average costs at full dispatch:

$$
\begin{aligned}
\hat{c}_1 &= c_1 + F_1 / K_1 \\
\hat{c}_2 &= F_2 / K_2 .
\end{aligned}
$$

In other words, since there is a single step in the variable cost, the problem reduces to the same formulation as the dispatchable approximation. Then the convex hull of the total minimum cost function can be found as:

$$
\begin{aligned}
& v^*(y) = \underset{x_1, x_2}{Min} \quad \hat{c}_1 x_1 + \hat{c}_2 x_2 \\
& s.t. \\
& 0 \leq x_1 \leq K_1 \\
& 0 \leq x_2 \leq K_2 \\
& x_1 + x_2 = y.
\end{aligned}
$$

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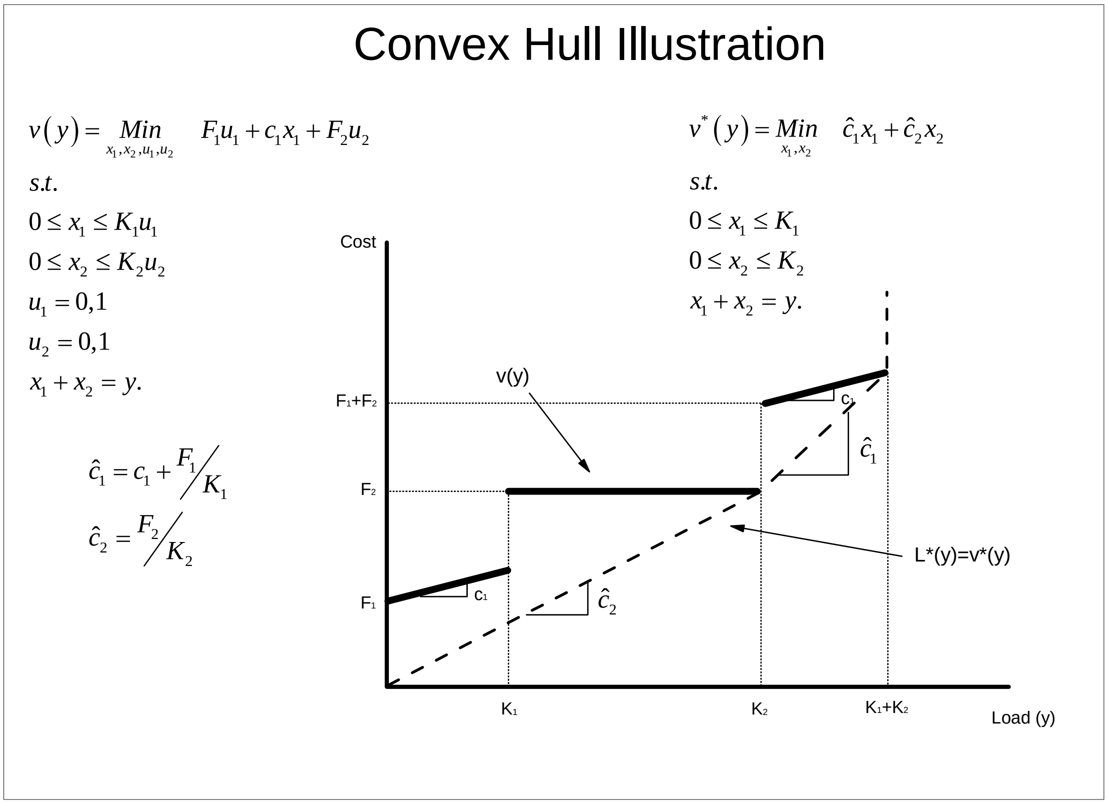
[21] Arthur M. Geoffrion, "Elements of Large-Scale Mathematical Programming, Parts I and II," Management Science, Vol. 16, No. 11, July 1970, pp. 652-691.



Note that the implicit commitment and dispatch with v_i^* is not the same as the commitment and dispatch with v_i. But the convex hull equals the dual solution objective function and the dual prices are everywhere equivalent to the slopes of the convex hull.

Applying the above algorithm and approximations is trivial in this case and solves the problem in one pass because the linear support defined here by the average cost is the complete convex hull for each function. In general, there may be pieces in the component convex hulls, and more than one pass would be required.

Electricity Market Model

The arguments above specialize to the electricity unit commitment and economic dispatch problem. For notational simplicity, the formulation here assumes that aggregate demand is fixed and the focus is on the economic commitment and dispatch of generation over a short horizon of a few interconnected periods. Operating reserve requirements are not represented. Further, generators are treated as a single representative generator at each node in the grid, having the same index as the node. Representing multiple generators, demand bids, operating reserves and simultaneous determination of energy and reserve prices raises no fundamental issues but would complicate the notation.

Introduction of multiple periods addresses the dynamics over the commitment period. Such intertemporal models always present questions about initial and ending conditions. For example, it may be that some units are on line and still operating based on a previous commitment that cannot be changed. There may be an obligation to make uplift payments for these units, but the decisions are fixed, the costs are sunk and the commitment decision is not a choice in the prospective unit commitment problem formulation.[22] Hence, these units are not part of the uplift as treated here.

The stylized version of the unit commitment and dispatch problem is formulated as:

$$\inf_{\text{g,d,on,start}} \sum_t \sum_i \left(StartCost_{it} \cdot start_{it} + NoLoad_{it} \cdot on_{it} + GenCost_{it}\left(g_{it}\right)\right)$$

subject to

$$\begin{array}{ll}
m_{it} \cdot on_{it} \le g_{it} \le M_{it} \cdot on_{it} & \forall i,t \\
-ramp_{it} \le g_{it} - g_{i,t-1} \le ramp_{it} & \forall i,t \\
start_{it} \le on_{it} \le start_{it} + on_{i,t-1} & \forall i,t \\
start_{it} = 0 \text{ or } 1 & \forall i,t \\
on_{it} = 0 \text{ or } 1 & \forall i,t \\
\mathbf{e}^T\left(\mathbf{g}_t - \mathbf{d}_t\right) - LossFn_t\left(\mathbf{d}_t - \mathbf{g}_t\right) = 0 & \forall t \\
Flow_{kt}\left(\mathbf{g}_t - \mathbf{d}_t\right) \le \overline{F}_{kt}^{\max} & \forall k,t \\
\mathbf{d}_t = \mathbf{y}_t & \forall t.
\end{array}$$

Indices:

nodes i (and unit at node)

time periods t

transmission constraints k.

Variables:

$$start_{it} = \begin{cases} 0 \text{ if unit i is not started in period t} \\ 1 \text{ if unit i is started in period t} \end{cases}$$

$$on_{it} = \begin{cases} 0 \text{ if unit i is off in period t} \\ 1 \text{ if unit i is on in period t} \end{cases}$$

$g_{it} =$ output of unit i in period t

$\mathbf{d}_t =$ vector of nodal demands in period t.

Constants:

[22] Scott Harvey, private communication.

$\mathbf{y}_t =$ vector of nodal loads in period t
m_{it} = minimum output from unit i in period t if unit is on
M_{it} = maximum output from unit i in period t if unit is on
$ramp_{it}$ = maximum ramp from unit i between period t-1 and period t
$StartCost_{it}$ = Cost to start unit i in period t
$NoLoad_{it}$ = No load cost for unit i in period t if unit is on
$\overline{F}_{kt}^{\max}$ = Maximum flow on transmission constraint k in period t.

Functions:
$GenCost_{it}(\cdot) =$ Production cost above No Load Cost to produce energy from unit i in period t
$LossFn_t(\cdot) =$ Losses in period t as a function of net nodal withdrawals
$Flow_{kt}(\cdot) =$ Flow on constraint k in period t as a function of net nodal injections.

In the notation of the value function description above, $f(x)$ is the objective function, x is all the variables, y is the vector or nodal loads for each period $\{\mathbf{y}_t\}$, the constraint $g(x) = y$ reduces to

$$\left\{\mathbf{e}^T(\mathbf{g}_t - \mathbf{d}_t) - LossFn_t(\mathbf{d}_t - \mathbf{g}_t)\right\} = 0$$
$$\left\{Flow_{kt}(\mathbf{g}_t - \mathbf{d}_t)\right\} \le \left\{\overline{F}_{kt}^{\max}\right\}$$
$$\{\mathbf{d}_t\} = \{\mathbf{y}_t\}.$$

The remaining constraints define the set X, and the optimal solution value of the objective is $v(y)$. The dual variables associated with the final constraints define the prices of interest.

Dispatch Based Approximation

A standard practice in the dispatch models is to linearize the transmission flow functions about given generation and load vectors: $\mathbf{g}'_t,\ \mathbf{d}'_t$

$$Flow_{kt}\left(\mathbf{g}_t-\mathbf{d}_t\right)\approx Flow_{kt}\left(\mathbf{g}'_t-\mathbf{d}'_t\right)+\left(\nabla Flow_{kt}\left(\mathbf{g}'_t-\mathbf{d}'_t\right)\right)^T\left(\mathbf{g}_t-\mathbf{d}_t-\left(\mathbf{g}'_t-\mathbf{d}'_t\right)\right)$$

$$Flow_{kt}\left(\mathbf{g}'_t-\mathbf{d}'_t\right)+\left(\nabla Flow_{kt}\left(\mathbf{g}'_t-\mathbf{d}'_t\right)\right)^T\left(\mathbf{g}_t-\mathbf{d}_t-\left(\mathbf{g}'_t-\mathbf{d}'_t\right)\right)\le \bar{F}_{kt}^{\max}$$

then $\left(\nabla Flow_{kt}\left(\mathbf{g}'_t-\mathbf{d}'_t\right)\right)^T\left(\mathbf{g}_t-\mathbf{d}_t\right)\le \bar{F}_{kt}^{\max}-Flow_{kt}\left(\mathbf{g}'_t-\mathbf{d}'_t\right)+\left(\nabla Flow_{kt}\left(\mathbf{g}'_t-\mathbf{d}'_t\right)\right)^T\left(\mathbf{g}'_t-\mathbf{d}'_t\right).$

Defining $F_{kt}^{\max}=\bar{F}_{kt}^{\max}-Flow_{kt}\left(\mathbf{g}'_t-\mathbf{d}'_t\right)+\left(\nabla Flow_{kt}\left(\mathbf{g}'_t-\mathbf{d}'_t\right)\right)^T\left(\mathbf{g}'_t-\mathbf{d}'_t\right).$

Then $\left(\nabla Flow_{kt}\left(\mathbf{g}'_t-\mathbf{d}'_t\right)\right)^T\left(\mathbf{g}_t-\mathbf{d}_t\right)\le F_{kt}^{\max}.$

Assuming that we are operating in a range where the voltage angle differences are small, we will have that

$-Flow_{kt}\left(\mathbf{g}'_t-\mathbf{d}'_t\right)+\left(\nabla Flow_{kt}\left(\mathbf{g}'_t-\mathbf{d}'_t\right)\right)^T\left(\mathbf{g}'_t-\mathbf{d}'_t\right)\approx 0$ or $F_{kt}^{\max}\approx \bar{F}_{kt}^{\max}$.

Given that the economic dispatch identifies a proposed solution, linearize the loss function about given generation and load vectors: $\mathbf{g}''_t,\ \mathbf{d}''_t$

$$LossFn_t\left(\mathbf{d}_t-\mathbf{g}_t\right)\approx LossFn_t\left(\mathbf{d}''_t-\mathbf{g}''_t\right)+\left(\nabla LossFn_t\left(\mathbf{d}''_t-\mathbf{g}''_t\right)\right)^T\left(\mathbf{d}_t-\mathbf{g}_t-\left(\mathbf{d}''_t-\mathbf{g}''_t\right)\right)$$

$$=\left(\nabla LossFn_t\left(\mathbf{d}''_t-\mathbf{g}''_t\right)\right)^T\left(\mathbf{d}_t-\mathbf{g}_t\right)-\left(\left(\nabla LossFn_t\left(\mathbf{d}''_t-\mathbf{g}''_t\right)\right)^T\left(\mathbf{d}''_t-\mathbf{g}''_t\right)-LossFn_t\left(\mathbf{d}''_t-\mathbf{g}''_t\right)\right)$$

Defining $\mathbf{LossSen}_t=\nabla LossFn_t\left(\mathbf{d}''_t-\mathbf{g}''_t\right)$,

and $OffSet_t=\left(\left(\nabla LossFn_t\left(\mathbf{d}''_t-\mathbf{g}''_t\right)\right)^T\left(\mathbf{d}''_t-\mathbf{g}''_t\right)-LossFn_t\left(\mathbf{d}''_t-\mathbf{g}''_t\right)\right)$,

then $LossFn_t\left(\mathbf{d}_t-\mathbf{g}_t\right)\approx \mathbf{LossSen}_t^T\left(\mathbf{d}_t-\mathbf{g}_t\right)-OffSet_t$.

We can re-write the approximation of the economic unit commitment and dispatch with the linearized functions

$$v\left(0,\left\{F_{kt}^{\max}\right\},\left\{\mathbf{y}_t\right\}\right)\equiv$$

$$\inf_{\text{g,d,on,start}}\sum_t\sum_i\left(StartCost_{it}\cdot start_{it}+NoLoad_{it}\cdot on_{it}+GenCost_{it}\left(g_{it}\right)\right)$$

subject to

$$\begin{aligned}
&m_{it}\cdot on_{it}\le g_{it}\le M_{it}\cdot on_{it} && \forall i,t\\
&-ramp_{it}\le g_{it}-g_{i,t-1}\le ramp_{it} && \forall i,t\\
&start_{it}\le on_{it}\le start_{it}+on_{i,t-1} && \forall i,t\\
&start_{it}=0 \text{ or } 1 && \forall i,t\\
&on_{it}=0 \text{ or } 1 && \forall i,t\\
&\mathbf{e}^T\left(\mathbf{g}_t-\mathbf{d}_t\right)+\mathbf{LossSen}_t^T\mathbf{g}_t-\mathbf{LossSen}_t^T\mathbf{d}_t+OffSet_t=0 && \forall t\\
&\nabla Flow_{kt}^T\left(\mathbf{g}_t-\mathbf{d}_t\right)\le F_{kt}^{\max} && \forall k,t\\
&\mathbf{d}_t=\mathbf{y}_t && \forall t
\end{aligned}$$

In the above, we suppressed $\mathbf{g}_t^{\prime}$, $\mathbf{d}_t^{\prime}$ when writing $\nabla Flow_{kt}\left(\mathbf{g}_t^{\prime}-\mathbf{d}_t^{\prime}\right)$ and write $\nabla Flow_{kt}$

If the value function v were convex as in the core model, the associated dual variables for the last three complicating constraints correspond to the prices and have an interpretation as the system marginal cost of energy, the marginal cost of transmission congestion and the locational marginal cost of energy to include the effect of congestion and transmission.

Unfortunately, the unit commitment problem value is not a convex function in general, as illustrated above.

Restricted Pricing Model

The corresponding version of the restricted price model takes the proposed optimal commitment decisions $\left\{start_{it}^{*}, on_{it}^{*}\right\}$ as given and restricts the solution to match this commitment.

$$
\begin{aligned}
& v^r\left(0,\left\{F_{kt}^{\max}\right\},\left\{\mathbf{y}_t\right\}\right)\equiv \\
& \inf_{\text{g,d,on,start}} \sum_t \sum_i \left(StartCost_{it}\cdot start_{it} + NoLoad_{it}\cdot on_{it} + GenCost_{it}\left(g_{it}\right)\right) \\
& \text{subject to} \\
& m_{it}\cdot on_{it} \le g_{it} \le M_{it}\cdot on_{it} && \forall i,t \\
& -ramp_{it} \le g_{it} - g_{i,t-1} \le ramp_{it} && \forall i,t \\
& start_{it} \le on_{it} \le start_{it} + on_{i,t-1} && \forall i,t \\
& start_{it} = start_{it}^{*} && \forall i,t \\
& on_{it} = on_{it}^{*} && \forall i,t \\
& \mathbf{e}^T\left(\mathbf{g}_t-\mathbf{d}_t\right)+\mathbf{LossSen}_t^T\mathbf{g}_t-\mathbf{LossSen}_t^T\mathbf{d}_t+OffSet_t=0 && \forall t \\
& \nabla Flow_{kt}^T\left(\mathbf{g}_t-\mathbf{d}_t\right)\le F_{kt}^{\max} && \forall k,t \\
& \mathbf{d}_t=\mathbf{y}_t && \forall t.
\end{aligned}
$$

This is a special case of a dispatch problem that can be solved using normal economic dispatch software.

Dispatchable Pricing Model

The dispatchable pricing model relaxes the integer requirements for discrete zero-one representation of the unit commitment decisions. The relaxation model treats these as continuous variables between zero and one.

$$
\begin{aligned}
& v^d\left(0,\left\{F_{kt}^{\max}\right\},\left\{\mathbf{y}_t\right\}\right)\equiv \\
& \inf_{\text{g,d,on,start}} \sum_t \sum_i \left(StartCost_{it}\cdot start_{it} + NoLoad_{it}\cdot on_{it} + GenCost_{it}\left(g_{it}\right)\right) \\
& \text{subject to} \\
& m_{it}\cdot on_{it} \le g_{it} \le M_{it}\cdot on_{it} && \forall i,t \\
& -ramp_{it} \le g_{it} - g_{i,t-1} \le ramp_{it} && \forall i,t \\
& start_{it} \le on_{it} \le start_{it} + on_{i,t-1} && \forall i,t \\
& 0 \le start_{it} \le 1 && \forall i,t \\
& 0 \le on_{it} \le 1 && \forall i,t \\
& \mathbf{e}^T\left(\mathbf{g}_t - \mathbf{d}_t\right) + \mathbf{LossSen}_t^T\mathbf{g}_t - \mathbf{LossSen}_t^T\mathbf{d}_t + OffSet_t = 0 && \forall t \\
& \nabla Flow_{kt}^T\left(\mathbf{g}_t - \mathbf{d}_t\right) \le F_{kt}^{\max} && \forall k,t \\
& \mathbf{d}_t = \mathbf{y}_t && \forall t.
\end{aligned}
$$

This is a different type of special case of a dispatch problem that can be solved using normal economic dispatch software.

Convex Hull, Minimum Uplift, Dual Pricing Model

The convex hull pricing model that minimizes the uplift corresponds to the dual solution. In the core model this is the same as the value function. In the general case the convex hull solution can be different. We form a dual optimization problem by dualizing or pricing the power balance equation and flow constraints into the objective function.

$$
\begin{aligned}
&v^h\left(0,\left\{F_{kt}^{\max}\right\},\left\{\mathbf{y}_t\right\}\right)\equiv \\
&\sup_{p,\lambda,\mu}\left\{\begin{aligned}
&-\sum_t \lambda_t OffSet_t-\sum_t\sum_k \mu_{kt}F_{kt}^{\max}+\sum_t \mathbf{p}_t^T\mathbf{y}_t \\
&+\left\{\begin{aligned}
&\inf_{\text{g,d,on,start}}\left(\begin{aligned}&\sum_t\sum_i\left(StartCost_{it}\cdot start_{it}+NoLoad_{it}\cdot on_{it}+GenCost_{it}\left(g_{it}\right)\right)-\sum_t\lambda_t\mathbf{e}^T\left(\mathbf{g}_t-\mathbf{d}_t\right)\\ &-\sum_t\lambda_t\cdot\mathbf{LossSen}_t^T\left(\mathbf{g}_t-\mathbf{d}_t\right)+\sum_t\sum_k\mu_{kt}\nabla Flow_{kt}^T\left(\mathbf{g}_t-\mathbf{d}_t\right)-\sum_t\mathbf{p}_t^T\mathbf{d}_t\end{aligned}\right)\\
&\text{subject to}\\
&m_{it}\cdot on_{it}\le g_{it}\le M_{it}\cdot on_{it} && \forall i,t\\
&-ramp_{it}\le g_{it}-g_{i,t-1}\le ramp_{it} && \forall i,t\\
&start_{it}\le on_{it}\le start_{it}+on_{i,t-1} && \forall i,t\\
&start_{it}=0\text{ or }1 && \forall i,t\\
&on_{it}=0\text{ or }1 && \forall i,t
\end{aligned}\right\}
\end{aligned}\right\}
\end{aligned}
$$

subject to

$$\mu_{kt}\ge 0 \qquad \forall t.$$

By inspection, we see that the inner problem is unbounded unless the prices satisfy the relation

$$\mathbf{p}_t=\lambda_t\mathbf{e}+\lambda_t\cdot\mathbf{LossSen}_t-\sum_k\mu_{kt}\nabla Flow_{kt}.$$

Therefore, an equivalent restatement of the dual problem that is more transparent would be

$$
\begin{aligned}
&v^h\left(0,\left\{F_{kt}^{\max}\right\},\left\{\mathbf{y}_t\right\}\right) \equiv \\
&-\inf_{p,\lambda,\mu}\left\{\begin{array}{l}
\sum_t \lambda_t OffSet_t + \sum_t \sum_k \mu_{kt} F_{kt}^{\max} - \sum_t \mathbf{p}_t^T \mathbf{y}_t \\
+\left\{\begin{array}{ll}
\sup\limits_{\mathrm{g,on,start}} \left(\sum_t \mathbf{p}_t^T \mathbf{g}_t - \sum_t \sum_i \left(StartCost_{it} \cdot start_{it} + NoLoad_{it} \cdot on_{it} + GenCost_{it}\left(g_{it}\right)\right)\right) & \\
\text{subject to} & \\
m_{it} \cdot on_{it} \le g_{it} \le M_{it} \cdot on_{it} & \forall i,t \\
-ramp_{it} \le g_{it} - g_{i,t-1} \le ramp_{it} & \forall i,t \\
start_{it} \le on_{it} \le start_{it} + on_{i,t-1} & \forall i,t \\
start_{it} = 0 \text{ or } 1 & \forall i,t \\
on_{it} = 0 \text{ or } 1 & \forall i,t
\end{array}\right\}
\end{array}\right\} \\
&\text{subject to} \\
&\mu_{kt} \ge 0 \qquad \forall t \\
&\mathbf{p}_t = \lambda_t \mathbf{e} + \lambda_t \cdot \mathbf{LossSen}_t - \sum_k \mu_{kt} \nabla Flow_{kt} \qquad \forall t.
\end{aligned}
$$

This equivalent formulation would be useful for designing a computational procedure. However, for the connection to uplift, another equivalent formulation presents the minimum uplift version of the dual problem or convex hull model:

$$
\begin{aligned}
&MinUplift = v\left(0,\left\{F_{kt}^{\max}\right\},\left\{\mathbf{y}_t\right\}\right) - v^h\left(0,\left\{F_{kt}^{\max}\right\},\left\{\mathbf{y}_t\right\}\right) = \\
&\inf_{p,\lambda,\mu}\left\{\begin{array}{l}
v\left(0,\left\{F_{kt}^{\max}\right\},\left\{\mathbf{y}_t\right\}\right) + \sum_t \lambda_t OffSet_t + \sum_t \sum_k \mu_{kt} F_{kt}^{\max} - \sum_t \mathbf{p}_t^T \mathbf{y}_t \\
+\left\{\begin{array}{ll}
\sup\limits_{\mathrm{g,on,start}} \left(\sum_t \mathbf{p}_t^T \mathbf{g}_t - \sum_t \sum_i \left(StartCost_{it} \cdot start_{it} + NoLoad_{it} \cdot on_{it} + GenCost_{it}\left(g_{it}\right)\right)\right) & \\
\text{subject to} & \\
m_{it} \cdot on_{it} \le g_{it} \le M_{it} \cdot on_{it} & \forall i,t \\
-ramp_{it} \le g_{it} - g_{i,t-1} \le ramp_{it} & \forall i,t \\
start_{it} \le on_{it} \le start_{it} + on_{i,t-1} & \forall i,t \\
start_{it} = 0 \text{ or } 1 & \forall i,t \\
on_{it} = 0 \text{ or } 1 & \forall i,t
\end{array}\right\}
\end{array}\right\} \\
&\text{subject to} \\
&\mu_{kt} \ge 0 \qquad \forall t \\
&\mathbf{p}_t = \lambda_t \mathbf{e} + \lambda_t \cdot \mathbf{LossSen}_t - \sum_k \mu_{kt} \nabla Flow_{kt} \qquad \forall t.
\end{aligned}
$$

Given the prices, the interior problem separates into individual plant commitment and dispatch problems. This is the Lagrangian relaxation interpretation for the dual problem. The value function is convex in its arguments. The dual objective is concave in the prices and lends itself to a number of solution procedures developed for the Lagrangian relaxation.[23]

Denote the solution to the primal problem by $\left[\{g_{it}^*\},\{start_{it}^*\},\{on_{it}^*\}\right]$. The solution to the primal problem will satisfy:

$$\mathbf{e}^T\left(\mathbf{g}_t^* - \mathbf{d}_t^*\right) + \mathbf{LossSen}_t^*\left(\mathbf{g}_t^* - \mathbf{d}_t^*\right) + OffSet_t = 0$$

$$\mathbf{d}_t^* = \mathbf{y}_t$$

Using these relations, we can write the minimum uplift as:

$$MinUplift =$$

$$\inf_{p,\lambda,\mu}\left\{\begin{aligned}&\sum_i\left[\begin{pmatrix}\sup\limits_{g,on,start}\sum_t\left(p_{it}g_{it} - StartCost_{it}\cdot start_{it} - NoLoad_{it}\cdot on_{it} - GenCost_{it}\left(g_{it}\right)\right)\\ \text{subject to}\\ m_{it}\cdot on_{it} \le g_{it} \le M_{it}\cdot on_{it} \qquad \forall i,t\\ -ramp_{it} \le g_{it} - g_{i,t-1} \le ramp_{it} \qquad \forall i,t\\ start_{it} \le on_{it} \le start_{it} + on_{i,t-1} \qquad \forall i,t\\ start_{it} = 0 \text{ or } 1 \qquad \forall i,t\\ on_{it} = 0 \text{ or } 1 \qquad \forall i,t\end{pmatrix} - \left(\sum_t\left(p_{it}g_{it}^* - StartCost_{it}\cdot start_{it}^* - NoLoad_{it}\cdot on_{it}^* - GenCost_{it}\left(g_{it}^*\right)\right)\right)\right]\\ &+\left[\sum_t\sum_k\left(\mu_{kt}F_{kt}^{\max} - \mu_{kt}\nabla Flow_{kt}^T\left(\mathbf{g}_t^* - \mathbf{d}_t^*\right)\right)\right]\end{aligned}\right\}$$

subject to

$$\mu_{kt} \ge 0 \qquad \forall k,t$$

$$\mathbf{p}_t = \lambda_t\mathbf{e} + \lambda_t\mathbf{LossSen}_t - \sum_k \mu_{kt}\nabla Flow_{kt} \qquad \forall t$$

The first terms in the above uplift are the differences between the optimal profits for each generating unit given the proposed price and the actual profits at the proposed solution. The last terms are the difference between the implied value of available transmission capacity at the proposed prices and the value of the flows on the transmission at the proposed solution. Depending on the configuration of transmission rights, this could be the difference the payments that the RTO may have to make to

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[23] L. Dubost, R. Gonzalez, C. Lemarechal, " A Primal-Proximal Heuristic Applied to the French Unit-Commitment Problem," Mathematical Programming, Series A, Vol. 104, 2005, pp. 129-151.

holder of financial transmission rights (FTRs) and the congestion charges that it collects from flows in the market.[24]

At the proposed solution, we have $\nabla Flow_{kt}^{T}\left(\mathbf{g}_t^* - \mathbf{d}_t^*\right) \le F_{kt}^{\max}$. Consequently, the FTR payment obligation could exceed the congestion charges collected by the RTO. This can be viewed as another uplift. How such a shortfall is handled varies among the RTOs.

The flow on a constraint at the optimal solution to the primal problem can be below the constraint limit while the dual assigns a nonzero shadow price to the constraint. One way in which this can happen is if the system operator must commit a unit to enforce a constraint but the minimum output of the unit would cause the flow on the constraint to drop below the limit.

We note that this approach to setting prices can be applied to minimize required uplifts even if the proposed commitment and schedule is not the minimum cost solution.

Extensions and Computational Tests

The simple examples above illustrate the basic properties of the alternative price models. Preliminary tests of simple models with multiple locations, network constraints, multiple periods, ramping limits, and demand bids produce results with pricing properties that are similar to the simple graphical illustrations. The equivalence of minimum uplift prices with the prices obtained from the Lagrangian relaxation, even in the presence of a duality gap, provides a rich source of experience about the behavior of such prices.[25] The need for alternative pricing models arises because of the duality gap. It is known that the relative magnitude of the duality gap, and hence the minimum uplift, decreases as the size of the problem increases. In other words, as the number of plants with material fixed costs increases, there are more choices and discontinuities at the point of a change in the commitment decision are less pronounced.[26]

The presence of transmission contingency constraints raises the question of how many of the constraints can be defined as "non-limiting" and dropped from the dispatched-based pricing model. Further simplifications and specializations of each model would be available depending on the information that would be available from the economic unit commitment and dispatch software. Although the theory establishes the convex hull approximation as the dual solution and this minimizes the uplift, it is not clear how large the differences in prices and uplift would be across the three price models

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24 With financial transmission rights, the worst case would be if all the constraints with positive shadow prices were binding in the allocation of FTRs, in which case the implied transmission payment is the congestion revenue deficiency.

25 For example, see A. Borghetti, G. Gross, C. A. Nucci, "Auctions with Explicit Demand-Sied Bidding in Competitive Electricity Markets," in Benjamin J. Hobbs, Michael H. Rothkopf, Richard P. O'Neill, Hung-po Chao (eds.), The Next Generation of Electric Power Unit Commitment Models, Kluwer Academic Publishers, Boston, pp. 53-74.

26 L.A.F.M. Ferreira, "On the Duality Gap for Thermal Unit Commitment Problems," IEEE International Symposium on Circuits and Systems, Volume 4, May 3-6, 1993 Page(s):2204 – 2207. Steven Stoft, Power System Economics, Wiley-Interscience, 2002, pp. 300-302.

in realistic applications. The relative magnitude might affect the workability of the assumption that the short-run incentive effects of uplift payments are de minimus.[27]

The interaction between day-ahead and real-time prices is another area to address. In the core model with risk neutrality there is a natural connection with expected real-time prices approximately equal to day-ahead prices. With the introduction of uplift, the formal connection would presumably be more complicated. As Ring[28] and Stoft[29] point out, the full long run incentive effects of these pricing rules to include the complications of multi-part bids, market power, and investment are not well understood. The magnitude of the differences is an empirical question that would be addressed through sensitivity analysis of realistic problems with the full array of plants, offers, loads, bids, reliability commitments, and transmission constraints.

Summary

Electricity market models require energy prices for balancing, spot and short-term forward transactions. For the simplest version of the core economic dispatch problem, the formulation produces a well-defined solution to the pricing problem in the usual intersection of the supply marginal cost and the demand bids. This pricing supports the equilibrium solution and satisfies a no arbitrage condition. In the more general economic unit commitment and dispatch models, there may be no corresponding uniform energy price vector that supports the solution. This introduces a need both define the appropriate energy prices and determine the associated uplift make-whole payments needed to support the solution. Different approximation of the optimal value function yield different price and uplift results. Simple examples illustrate the differences. Examination of the relative magnitudes of the differences would require practical computational testing.

Endnotes

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[27] The incentive to manipulate uplift payments by changing startup cost offers is balanced by the relative ease of monitoring these costs and Regional Transmission Organizations can require these offers to remain constant for long periods.

[28] Ring, 1995, p. 213.

[29] Stoft, 2002, p. 302.

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